Topological expansion of mixed correlations in the hermitian 2 Matrix Model and x-y symmetry of the F__g invariants.

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SPhT-T07/055

Topological expansion of mixed correlations in the hermitian 2 Matrix Model and x-y symmetry of the F_g algebraic invariants.

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Abstract: We compute expectation values of mixed traces containing both matrices in a two matrix model, i.e. generating function for counting bicolored discrete surfaces with non uniform boundary conditions. As an application, we prove the $x-y$ symmetry of [21].

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1 Introduction

Formal matrix integrals can be regarded as an efficient toy model to explore the link between algebraic geometry and integrable systems [31, 3]. The theory of quantum gravity [12, 11, 27] is based on the idea that matrix models provide a generating function to measure "volumes" of moduli spaces of Riemann surfaces, and random matrix models were introduced in the 80's [6] as a discretized version of 2d quantum gravity, i.e. conformal field theory coupled to gravity.

The formal matrix integral is at the same time a tau-function of some integrable hierarchy [12], and it has a 't Hoof topological expansion [33, 12, 1]:

$$\ln \int_{\text{formal}} dM \mathrm{e}^{-N\,\mathrm{Tr}\,V(M)} = \sum_{g=0}^{\infty} N^{2-2g} F^{(g)} \tag{1-1}$$

which is related to algebraic geometry (see [5, 3, 29, 14]).

In a recent work [13, 7, 19, 8, 21], we have developped a method to compute the $F^{(g)}$'s for various formal hermitian matrix models (1-matrix model, 2-matrix model, matrix model with an external field, double scaling limits of 2-matrix model) out of the data of an algebraic equation (called the classical spectral curve):

$$\mathcal{E}(x,y) = 0 \qquad , \qquad \mathcal{E} = \text{polynomial.} \tag{1-2}$$

The construction of [21] extends beyond matrix models, and the $F^{(g)}$'s can be computed for any algebraic equation of the type $\mathcal{E}(x,y) = 0$.

However the construction of [21] assumes an embedding of the curve into $\mathbf{C}^2$, i.e. the choice of 2 meromorphic functions x and y on the curve. It was claimed in [21] that $F^{(g)}$ is invariant under the exchange $x \leftrightarrow y$, and the proof was announced to be published separately.

This is what we do in the present paper, together with additional results.

Mixed correlations

In order to prove this claim, we first explore the case where the $F^{(g)}$'s come from a formal 2-matrix model (the symmetry $x \leftrightarrow y$ holds almost by definition in that case, see [8]). We write the loop equation relations (W-algebra) [32, 18], which we solve, and we are led to define new mixed correlation functions ($W_{k,l}$ and $H_{k,l}$ below), which did not appear in [21].

In the application of the 2-matrix model to quantum gravity and conformal field theory, those mixed correlation functions were known to play an important role in the understanding of boundary operators. But their explicit computation has been a challenge until recently. The main reason is that they don't reduce to eigenvalues of the matrices, and could not be computed by standard methods. The first explicit computations were obtained in [4] and [17]. Here in this paper, we show how to compute the topological expansion of a family of mixed correlation functions of the 2-matrix model. In a coming work [23], we shall show how to compute all mixed correlations, and introduce a link with group theory and Bethe ansatz (this is a generalization of [22]).

Then, for the general case (i.e. if $\mathcal{E}$ was not obtained from a matrix model), we mimic those mixed correlation functions and that allows to prove the $x \leftrightarrow y$ symmetry of $F^{(g)}$.

2 Mixed traces of matrix models

Consider the formal 2-matrix integral[3]:

$$Z = \int dM_1\, dM_2\; \mathrm{e}^{-N \operatorname{tr} (V_1(M_1)+V_2(M_2)-M_1 M_2)} \tag{2-1}$$

where we assume in this section that V_1 is a polynomial of degree d_1+1 and V_2 is a polynomial of degree d_2+1.

Our goal is to compute the following connected expectation values:

$$\begin{aligned}
& \overline{W}_{k,l}(x_1,\dots,x_k|y_1,\dots,y_l) \\
= & \left\langle \operatorname{tr}\frac{1}{x_1-M_1} \operatorname{tr}\frac{1}{x_2-M_1}\dots \operatorname{tr}\frac{1}{x_k-M_1}\;\; \operatorname{tr}\frac{1}{y_1-M_2} \operatorname{tr}\frac{1}{y_2-M_2}\dots \operatorname{tr}\frac{1}{y_l-M_2}\right\rangle_c \\
= & \sum_{g=0}^{\infty} N^{2-2g-k-l}\, \overline{W}^{(g)}_{k,l}(x_1,\dots,x_k|y_1,\dots,y_l).
\end{aligned} \tag{2-2}$$

and

$$\begin{aligned}
& \overline{H}_{k,l}(x,y;x_1,\dots,x_k|y_1,\dots,y_l) \\
= & \left\langle \operatorname{tr}\frac{1}{x-M_1}\frac{1}{y-M_2}\;\; \operatorname{tr}\frac{1}{x_1-M_1}\dots \operatorname{tr}\frac{1}{x_k-M_1}\;\; \operatorname{tr}\frac{1}{y_1-M_2}\dots \operatorname{tr}\frac{1}{y_l-M_2}\right\rangle_c \\
= & \sum_{g=0}^{\infty} N^{2-2g-k-l-1}\, \overline{H}^{(g)}_{k,l}(x,y;x_1,\dots,x_k|y_1,\dots,y_l)
\end{aligned} \tag{2-3}$$

$\overline{W}^{(g)}_{k,l}$ is the generating function which counts connected genus g bi-colored discrete surfaces with k boundaries of the first color, and l boundaries of the second color. $\overline{H}^{(g)}_{k,l}$ is the generating function which counts genus g bi-colored discrete surfaces with

[3]A formal integral is defined as a formal power series in some expansion parameter t, as explained in [20] or [21]. Formal matrix integrals always have a $1/N^2$ expansion order by order in t, called the topological expansion.

k boundaries of the first color, and l boundaries of the second color, and one additional boundary which carries the 2 colors. The power of N in both cases is the Euler characteristic of such surfaces. The 2-matrix model was introduced in [28] as a discrete version of the Ising model on a random surface.

Notice that in $\overline{H}_{k,l}^{(g)}$, the first trace contains both matrices M_1 and M_2, we call it a **mixed trace** because it cannot be expressed in terms of eigenvalues of M_1 and M_2. In applications of matrix models to conformal field theories, such objects correspond to the insertion of a pair of boundary operators, and are thus very interesting. $\overline{H}_{0,0}^{(0)}$ was computed in many works [18, 9], and in the context of convergent integrals (instead of formal integrals), $\overline{H}_{0,0}$ was computed in [4, 17, 2].

The $\overline{W}_{k,0}^{(g)}$'s were already computed in [13, 19, 8], and are given by the algebraic invariants defined in [21], they are the non mixed traces.

It is known (see for instance [8]) that all those functions are multivalued functions of their x or y variables, and they are in fact functions living on a Riemann surface called the spectral curve of equation:

$$\mathcal{E}(x,y) = 0. \tag{2-4}$$

On this curve, we chose a canonical basis of cycles[4] $\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}$, $i,j = 1,\dots \mathcal{G}$, where $\mathcal{G}$ denotes the genus of the curve $\mathcal{E}$. We will note by p^i (resp. $\tilde{p}^j$) the different points of $\mathcal{E}$ whose projection in the complex plane by the meromorphic function x (resp. y) are equal:

$$\forall i = 1 \dots d_2\,,\ x(p^i) = x(p^0) \qquad , \qquad \forall i = 1 \dots d_1\,,\ y(\tilde{p}^i) = x(\tilde{p}^0), \tag{2-5}$$

where the superscript 0 refers to the x- and y-physical sheets.

It is thus more convenient to redefine $\overline{W}_{k,l}^{(g)}$ and $\overline{H}_{k,l}^{(g)}$ in terms of meromorphic forms on the curve:

$$\begin{aligned}
& W_{k,l}^{(g)}(p_1,\dots,p_k|q_1,\dots,q_l) \\
= \quad & \overline{W}_{k,l}^{(g)}(x(p_1),\dots,x(p_k)|y(q_1),\dots,y(q_l))\ dx(p_1)\dots dx(p_k)dy(q_1)\dots dy(q_l) \\
& +\delta_{g,0}\delta_{k,1}\delta_{l,0}(y(p_1)-V_1'(x(p_1)))dx(p_1)+\delta_{g,0}\delta_{k,0}\delta_{l,1}(x(q_1)-V_2'(y(q_1)))dy(q_1) \\
& +\frac{\delta_{g,0}\delta_{k,2}\delta_{l,0}\ dx(p_1)dx(p_2)}{(x(p_1)-x(p_2))^2} \\
& +\frac{\delta_{g,0}\delta_{k,0}\delta_{l,2}\ dy(q_1)dy(q_2)}{(y(q_1)-y(q_2))^2}
\end{aligned}$$

(2 − 6)

[4] All required definitions relative to algebraic geometry can be found in [21] or more generally in [25, 24]. We will use all along these notes the notations of [21]. The $\mathcal{A}$ and $\mathcal{B}$-cycles may be the modified cycles of [21].

where the p_i's and q_j's are now points on the curve $\mathcal{E}$, instead of points in the complex plane. We have also "renormalized the unstable functions" with $2-2g-k-l \geq 0$.

With those notations we have [8, 5]:

$$W_{1,0}^{(0)} = W_{0,1}^{(0)} = 0, \tag{2-7}$$

$$W_{2,0}^{(0)}(p,q) = -W_{1,1}^{(0)}(p,q) = W_{0,2}^{(0)}(p,q) = B(p,q) \tag{2-8}$$

where B is the Bergmann kernel, i.e. the unique bilinear form on $\mathcal{E}$ with a double pole at $p=q$ and no other pole, with vanishing residue, and normalized on $\mathcal{A}$-cycles:

$$B(p,q) \underset{p\to q}{\sim} \frac{dz(p)dz(q)}{(z(p)-z(q))^2} + \text{finite} \qquad , \qquad \forall i = 1\dots\mathcal{G}\,,\ \oint_{\mathcal{A}} B = 0. \tag{2-9}$$

We also define the differentials corresponding to the mixed correlation functions:

$$\begin{aligned} & H_{k,l}^{(g)}(p,q;p_1,\dots,p_k|q_1,\dots,q_l) \\ = \ & \overline{H}_{k,l}^{(g)}(x(p),y(q);x(p_1),\dots,x(p_k)|y(q_1),\dots,y(q_l))\ dx(p_1)\dots dx(p_k)dy(q_1)\dots dy(q_l) \\ & +\delta_{g,0}\delta_{k,0}\delta_{l,0} \end{aligned} \tag{2-10}$$

and we normalize them by the leading order of the simplest mixed correlation function:

$$h_{k,l}^{(g)}(p,q;p_1,\dots,p_k|q_1,\dots,q_l) = \frac{H_{k,l}^{(g)}(p,q;p_1,\dots,p_k|q_1,\dots,q_l)}{H_{0,0}^{(0)}(p,q)}. \tag{2-11}$$

It is well known [14, 18, 9] (and it can be rederived from Eq. (2-18) and Eq. (2-21) below) that:

$$H_{0,0}^{(0)}(p,q) = \frac{\mathcal{E}(x(p),y(q))}{(x(p)-x(q))(y(p)-y(q))}. \tag{2-12}$$

We also need to introduce:

$$\begin{aligned} & U_{k,l}(p,y;p_1,\dots,p_k|q_1,\dots,q_l) \\ = \ & \Big\langle \operatorname{tr} \frac{1}{x(p)-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\ \operatorname{tr}\frac{dx(p_1)}{x(p_1)-M_1}\dots \operatorname{tr}\frac{dx(p_k)}{x(p_k)-M_1} \\ & \operatorname{tr}\frac{dy(q_1)}{y(q_1)-M_2}\dots \operatorname{tr}\frac{dy(q_l)}{y(q_l)-M_2}\Big\rangle_c \\ & +\delta_{g,0}\delta_{k,0}\delta_{l,0}(V_2'(y)-x(p)) \\ = \ & \sum_{g=0}^{\infty} N^{2-2g-k-l-1}\, U_{k,l}^{(g)}(p,y;p_1,\dots,p_k|q_1,\dots,q_l), \end{aligned} \tag{2-13}$$

which is a polynomial of y of degree at most d_2-1,

$$\begin{aligned} & \tilde{U}_{k,l}(x,q;p_1,\dots,p_k|q_1,\dots,q_l) \\ = \ & \Big\langle \operatorname{tr}\frac{V_1'(x)-V_1'(M_1)}{x-M_1}\frac{1}{y(q)-M_2}\ \operatorname{tr}\frac{dx(p_1)}{x(p_1)-M_1}\dots \operatorname{tr}\frac{dx(p_k)}{x(p_k)-M_1} \end{aligned}$$

$$
\begin{aligned}
& \mathrm{tr}\, \frac{dy(q_1)}{y(q_1)-M_2} \dots \mathrm{tr}\, \frac{dy(q_l)}{y(q_l)-M_2} \Big\rangle_c \\
& +\delta_{g,0}\delta_{k,0}\delta_{l,0}(V_1'(x)-y(p)) \\
= & \sum_{g=0}^{\infty} N^{2-2g-k-l-1}\, \tilde{U}_{k,l}^{(g)}(x,q;p_1,\dots,p_k|q_1,\dots,q_l),
\end{aligned} \tag{2-14}
$$

which is a polynomial of x of degree at most d_1-1 and

$$
\begin{aligned}
& -E_{k,l}(x,y;p_1,\dots,p_k|q_1,\dots,q_l) \\
= & \Big\langle \mathrm{tr}\, \frac{V_1'(x)-V_1'(M_1)}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\ \mathrm{tr}\, \frac{dx(p_1)}{x(p_1)-M_1} \dots \mathrm{tr}\, \frac{dx(p_k)}{x(p_k)-M_1} \\
& \mathrm{tr}\, \frac{dy(q_1)}{y(q_1)-M_2} \dots \mathrm{tr}\, \frac{dy(q_l)}{y(q_l)-M_2} \Big\rangle_c \\
& +\delta_{g,0}\delta_{k,0}\delta_{l,0}((V_1'(x)-y(p))(V_2'(y)-x(p))-1) \\
= & -\sum_{g=0}^{\infty} N^{2-2g-k-l-1}\, E_{k,l}^{(g)}(x,y;p_1,\dots,p_k|q_1,\dots,q_l),
\end{aligned} \tag{2-15}
$$

which is a polynomial of x of degree d_1-1 and of y of degree d_2-1.

We have:

$$
E_{0,0}^{(0)}(x,y) = \mathcal{E}(x,y) \qquad , \qquad U_{0,0}^{(0)}(p,y) = \frac{\mathcal{E}(x(p),y)}{y-y(p)} \qquad , \qquad \tilde{U}_{0,0}^{(0)}(x,q) = \frac{\mathcal{E}(x,y(q))}{x-x(q)}, \tag{2-16}
$$

and

$$
P_{0,0}^{(0)}(x,y) = -\mathcal{E}(x,y). \tag{2-17}
$$

2.1 Loop equations

In order to obtain a closed set of equations computing these mixed correlation functions, we consider 4 families of loop equations [32, 18, 16] corresponding to different infinitesimal changes of variables $M_i \to M_i + \epsilon\delta M_i$ in the matrix integral.

$\delta M_2 = \frac{1}{x(p)-M_1}\frac{1}{y(q)-M_2}\prod_{i=1}^{k}\mathrm{tr}\,\frac{1}{x(p_i)-M_1}\prod_{j=1}^{l}\mathrm{tr}\,\frac{1}{y(q_j)-M_2}$ gives:

$$
\begin{aligned}
-\,U_{k,l}^{(g)}(p,y(q);\mathbf{p_K}|\mathbf{q_L}) \;=\; & (x(p)-x(q))H_{k,l}^{(g)}(p,q;\mathbf{p_K}|\mathbf{q_L}) \\
& +\sum_h\sum_{I,J}\frac{W_{i,j+1}^{(h)}(\mathbf{p_I}|\mathbf{q_J},q)H_{k-i,l-j}^{(g-h)}(p,q;\mathbf{p_{K/I}}|\mathbf{q_{L/J}})}{dy(q)} \\
& +\frac{H_{k,l+1}^{(g-1)}(p,q;\mathbf{p_K}|\mathbf{q_L},q)}{dy(q)} \\
& -\sum_n d_{q_n}\,\frac{H_{k,l-1}^{(g)}(p,q_n;\mathbf{p_K}|\mathbf{q_{L/\{n\}}})}{y(q)-y(q_n)}
\end{aligned} \tag{2-18}
$$

$\delta M_1 = \frac{1}{x(p)-M_1}\frac{1}{y(q)-M_2}\prod_{i=1}^{k} \mathrm{tr}\, \frac{1}{x(p_i)-M_1}\prod_{j=1}^{l} \mathrm{tr}\, \frac{1}{y(q_j)-M_2}$ gives:

$$
\begin{aligned}
-\tilde{U}^{(g)}_{k,l}(x(p),q;\mathbf{p_K}|\mathbf{q_L}) &= (y(q)-y(p))H^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L}) \\
&\quad + \sum_h \sum_{I,J} \frac{W^{(h)}_{i+1,j}(p,\mathbf{p_I}|\mathbf{q_J})H^{(g-h)}_{k-i,l-j}(p,q;\mathbf{p_{K/I}}|\mathbf{q_{L/J}})}{dx(p)} \\
&\quad + \frac{H^{(g-1)}_{k+1,l}(p,q;p,\mathbf{p_K}|\mathbf{q_L})}{dx(p)} \\
&\quad - \sum_m d_{p_m}\, \frac{H^{(g)}_{k-1,l}(p_m,q;\mathbf{p_{K/\{m\}}}|\mathbf{q_L})}{x(p)-x(p_m)}
\end{aligned}
\tag{2-19}
$$

$\delta M_2 = \frac{V_1'(x(p))-V_1'(M_1)}{x(p)-M_1}\frac{1}{y(q)-M_2}\prod_{i=1}^{k} \mathrm{tr}\, \frac{1}{x(p_i)-M_1}\prod_{j=1}^{l} \mathrm{tr}\, \frac{1}{y(q_j)-M_2}$ gives:

$$
\begin{aligned}
E^{(g)}_{k,l}(x(p),y(q);\mathbf{p_K}|\mathbf{q_L}) &= (x(p)-x(q))\tilde{U}^{(g)}_{k,l}(x(p),q;\mathbf{p_K}|\mathbf{q_L}) \\
&\quad + \sum_h \sum_{I,J} \frac{\check{W}^{(h)}_{i,j+1}(\mathbf{p_I}|\mathbf{q_J},q)\tilde{U}^{(g-h)}_{k-i,l-j}(x(p),q;\mathbf{p_{K/I}}|\mathbf{q_{L/J}})}{dy(q)} \\
&\quad + \frac{\tilde{U}^{(g-1)}_{k,l+1}(x(p),q;\mathbf{p_K}|\mathbf{q_L},q)}{dy(q)} \\
&\quad - \sum_m d_{q_m}\, \frac{\tilde{U}^{(g)}_{k,l-1}(x(p),q_m;\mathbf{p_K}|\mathbf{q_{L/\{m\}}})}{y(q)-y(q_m)} \\
&\quad - \sum_m d_{p_m}\, H^{(g)}_{k-1,l}(p_m,q;\mathbf{p_{K/\{m\}}}|\mathbf{q_L})
\end{aligned}
\tag{2-20}
$$

and $\delta M_1 = \frac{1}{x(p)-M_1}\frac{V_2'(y(q))-V_2'(M_2)}{y(q)-M_2}\prod_{i=1}^{k} \mathrm{tr}\, \frac{1}{x(p_i)-M_1}\prod_{j=1}^{l} \mathrm{tr}\, \frac{1}{y(q_j)-M_2}$ gives:

$$
\begin{aligned}
E^{(g)}_{k,l}(x(p),y(q);\mathbf{p_K}|\mathbf{q_L}) &= (y(q)-y(p))U^{(g)}_{k,l}(p,y(q);\mathbf{p_K}|\mathbf{q_L}) \\
&\quad + \sum_h \sum_{I,J} \frac{W^{(h)}_{i+1,j}(p,\mathbf{p_I}|\mathbf{q_J})U^{(g-h)}_{k-i,l-j}(p,y(q);\mathbf{p_{K/I}}|\mathbf{q_{L/J}})}{dx(p)} \\
&\quad + \frac{U^{(g-1)}_{k+1,l}(p,y(q);p,\mathbf{p_K}|\mathbf{q_L})}{dx(p)} \\
&\quad - \sum_m d_{p_m}\, \frac{U^{(g)}_{k-1,l}(p_m,y(q);\mathbf{p_{K/\{m\}}}|\mathbf{q_L})}{x(p)-x(p_m)} \\
&\quad - \sum_m d_{q_m}\, H^{(g)}_{k,l-1}(p,q_m;\mathbf{p_K}|\mathbf{q_{L/\{m\}}}).
\end{aligned}
\tag{2-21}
$$

Those loop equations can be seen to be equivalent to W-algebra constraints [10, 12], or to a generalization of Tutte's equations for the combinatorics of discrete surfaces [34, 35].

2.2 Solution of loop equations

Theorem 2.1 *The solution of loop equations is such that:*

$$\begin{aligned}
& h_{k,l}^{(g)}(p,q;\mathbf{p_K}|\mathbf{q_L}) \\
= \quad & \mathop{\mathrm{Res}}_{r\to\tilde{q}^j,p,\mathbf{p_K}} \frac{1}{(x(p)-x(r))(y(r)-y(q))}\Big(h_{k+1,l}^{(g-1)}(r,q;r,\mathbf{p_K}|\mathbf{q_L}) \\
& +\sum_h\sum_{I\subset K}\sum_{J\subset L} W_{i+1,j}^{(h)}(r,\mathbf{p_I}|\mathbf{q_J})h_{k-i,l-j}^{(g-h)}(r,q;\mathbf{p_{K/I}}|\mathbf{q_{L/J}})\Big),
\end{aligned} \tag{2-22}$$

$$\begin{aligned}
& W_{k,l+1}^{(g)}(\mathbf{p_K}|\mathbf{q_L},q) \\
= \quad & \mathop{\mathrm{Res}}_{r\to\tilde{q}^j,\mathbf{p_K}} \frac{dy(q)}{(y(r)-y(q))}\Big(h_{k+1,l}^{(g-1)}(r,q;r,\mathbf{p_K}|\mathbf{q_L}) \\
& +\sum_h\sum_{I\subset K}\sum_{J\subset L} W_{i+1,j}^{(h)}(r,\mathbf{p_I}|\mathbf{q_J})h_{k-i,l-j}^{(g-h)}(r,q;\mathbf{p_{K/I}}|\mathbf{q_{L/J}})\Big).
\end{aligned} \tag{2-23}$$

where $\mathop{\mathrm{Res}}_{r\to\tilde{q}^j}$ *means that one takes the residues around all the points* $\tilde{q}^j\neq q$ *such that* $y(\tilde{q}^j)=y(q)$*.*

Given the initial conditions:

$$h_{0,0}^{(0)}=1 \qquad , \qquad W_{k,0}^{(g)}(p_1,\dots,p_k)=W_k^{(g)}(p_1,\dots,p_k)\Big|_{\mathcal{E}} \tag{2-24}$$

where $W_k^{(g)}(p_1,\dots,p_k)\Big|_{\mathcal{E}}$ is the function defined in [21], the above system is triangular and computes univocally any $h_{k,l}^{(g)}$ and $W_{k,l}^{(g)}$ in at most $k+l+\frac{g^2}{2}$ steps.

One easily proves by recursion on $2g+k+l$ that:

$$H_{k,l}^{(g)}(p,q;\mathbf{p_K}|\mathbf{q_L}) \text{ has poles } \left\{\begin{array}{l} \text{in } p=a,q,\mathbf{q_L} \\ \text{in } q=b,p,\mathbf{p_K} \\ \text{in } p_j=a,q,\mathbf{q_L} \\ \text{in } q_j=b,p,\mathbf{p_K} \end{array}\right. \tag{2-25}$$

and

$$W_{k,l}^{(g)}(\mathbf{p_K}|\mathbf{q_L}) \text{ has poles } \left\{\begin{array}{l} \text{in } p_j=a,\mathbf{q_L} \\ \text{in } q_j=b,\mathbf{p_K} \end{array}\right. \tag{2-26}$$

proof:

Since $\tilde{U}_{k,l}^{(g)}(x(p),q;\mathbf{p_K}|\mathbf{q_L})$ is a polynomial in $x(p)$ of degree at most d_1-2, it is given by the Lagrange interpolation formula:

$$\tilde{U}_{k,l}^{(g)}(x(p),q;\mathbf{p_K}|\mathbf{q_L}) \quad = \quad \tilde{U}_{0,0}^{(0)}(x(p),q)\sum_{j=1}^{d_1}\frac{\tilde{U}_{k,l}^{(g)}(x(\tilde{q}^j),q;\mathbf{p_K}|\mathbf{q_L})}{(x(p)-x(\tilde{q}^j))\tilde{U}_{0,0\ x}^{(0)}(x(\tilde{q}^j),q)}$$

$$= \qquad \tilde{U}^{(0)}_{0,0}(x(p),q) \sum_{j=1}^{d_1} \mathop{\mathrm{Res}}_{r\to\tilde{q}^j} \frac{\tilde{U}^{(g)}_{k,l}(x(\tilde{q}^j),q;\mathbf{p_K}|\mathbf{q_L})\,dx(r)}{(x(p)-x(r))\,\tilde{U}^{(0)}_{0,0}(x(r),q)}. \tag{2-27}$$

Then we replace $\tilde{U}^{(g)}_{k,l}(x(\tilde{q}^j),q;\mathbf{p_K}|\mathbf{q_L})$ by its value from the loop equation 2-19:

$$\begin{aligned}\tilde{U}^{(g)}_{k,l}(x(p),q;\mathbf{p_K}|\mathbf{q_L}) \;=\; & -\sum_{j=1}^{d_1} \mathrm{Res}_{\,r\to\tilde{q}^j} \frac{\tilde{U}^{(0)}_{0,0}(x(p),q)}{(x(p)-x(r))\,\tilde{U}^{(0)}_{0,0}(x(r),q)} \Big[\\ & \sum_h \sum_{I,J} W^{(h)}_{i+1,j}(r,\mathbf{p_I}|\mathbf{q_J}) H^{(g-h)}_{k-i,l-j}(r,q;\mathbf{p_{K/I}}|\mathbf{q_{L/J}}) \\ & + H^{(g-1)}_{k+1,l}(r,q;r,\mathbf{p_K}|\mathbf{q_L}) - \sum_m d_{p_m} \frac{H^{(g)}_{k-1,l}(p_m,q;\mathbf{p_{K/\{m\}}}|\mathbf{q_L})\,dx(r)}{x(r)-x(p_m)} \Big]\end{aligned} \tag{2-28}$$

Notice that the same residue computed at $r \to p$ gives the terms in the RHS of the loop equation 2-19, and therefore:

$$\begin{aligned} & (y(q)-y(p))H^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L}) \\ = \; & \mathop{\mathrm{Res}}_{r\to p,\tilde{q}^j} \frac{\tilde{U}^{(0)}_{0,0}(x(p),q)}{(x(p)-x(r))\,\tilde{U}^{(0)}_{0,0}(x(r),q)} \Big[\\ & \sum_h \sum_{I,J} W^{(h)}_{i+1,j}(r,\mathbf{p_I}|\mathbf{q_J}) H^{(g-h)}_{k-i,l-j}(r,q;\mathbf{p_{K/I}}|\mathbf{q_{L/J}}) \\ & + H^{(g-1)}_{k+1,l}(r,q;r,\mathbf{p_K}|\mathbf{q_L}) - \sum_m d_{p_m} \frac{H^{(g)}_{k-1,l}(p_m,q;\mathbf{p_{K/\{m\}}}|\mathbf{q_L})\,dx(r)}{x(r)-x(p_m)} \Big]. \end{aligned} \tag{2-29}$$

Moreover the last term $d_{p_m} \frac{H^{(g)}_{k-1,l}(p_m,q;\mathbf{p_{K/\{m\}}}|\mathbf{q_L})\,dx(r)}{x(r)-x(p_m)}$ can be computed explicitely:

$$\begin{aligned} & d_{p_m}\, \mathrm{Res}_{\,r\to p,\tilde{q}^j} \frac{\tilde{U}^{(0)}_{0,0}(x(p),q)}{(x(p)-x(r))\,\tilde{U}^{(0)}_{0,0}(x(r),q)} \frac{H^{(g)}_{k-1,l}(p_m,q;\mathbf{p_{K/\{m\}}}|\mathbf{q_L})\,dx(r)}{x(r)-x(p_m)} \\ & = d_{p_m}\, \mathrm{Res}_{\,r\to p,\tilde{q}^j} \frac{\mathcal{E}(x(p),y(q))(x(r)-x(q))}{(x(p)-x(r))(x(p)-x(q))\,\mathcal{E}(x(r),y(q))} \frac{H^{(g)}_{k-1,l}(p_m,q;\mathbf{p_{K/\{m\}}}|\mathbf{q_L})\,dx(r)}{x(r)-x(p_m)}. \end{aligned} \tag{2-30}$$

Under this form, one can see that the integrant is a rational function of $x(r)$. Thus, the residue can be computed on the complex plane obtained by the projection x and we can move the integration contours on the complex plane instead of the curve $\mathcal{E}$ itself. This term is then equal to:

$$\begin{aligned} & d_{p_m} \mathop{\mathrm{Res}}_{x\to x(p),x(\tilde{q}^j)} \frac{\mathcal{E}(x(p),y(q))(x-x(q))}{(x(p)-x)(x(p)-x(q))\,\mathcal{E}(x,y(q))} \frac{H^{(g)}_{k-1,l}(p_m,q;\mathbf{p_{K/\{m\}}}|\mathbf{q_L})\,dx}{x-x(p_m)} \\ = \; & -d_{p_m} \mathop{\mathrm{Res}}_{x\to x(p_m)} \frac{\mathcal{E}(x(p),y(q))(x-x(q))}{(x(p)-x)(x(p)-x(q))\,\mathcal{E}(x,y(q))} \frac{H^{(g)}_{k-1,l}(p_m,q;\mathbf{p_{K/\{m\}}}|\mathbf{q_L})\,dx}{x-x(p_m)} \\ = \; & -d_{p_m} \frac{\mathcal{E}(x(p),y(q))(x(p_m)-x(q))}{(x(p)-x(p_m))(x(p)-x(q))\,\mathcal{E}(x(p_m),y(q))} H^{(g)}_{k-1,l}(p_m,q;\mathbf{p_{K/\{m\}}}|\mathbf{q_L}) \end{aligned}$$

$$
\begin{aligned}
&= -\operatorname*{Res}_{r\to p_m} \frac{\tilde{U}^{(0)}_{0,0}(x(p),q)}{(x(p)-x(r))\,\tilde{U}^{(0)}_{0,0}(x(r),q)}\, H^{(g)}_{k-1,l}(r,q;\mathbf{p}_{\mathbf{K}/\{\mathbf{m}\}}|\mathbf{q}_{\mathbf{L}}) W^{(0)}_{2,0}(r,p_m) \\
&= -\operatorname*{Res}_{r\to p_m} \frac{\tilde{U}^{(0)}_{0,0}(x(p),q)}{(x(p)-x(r))\,\tilde{U}^{(0)}_{0,0}(x(r),q)} \Big(\sum_{h,I,J} W^{(h)}_{i+1,j}(r,\mathbf{p}_{\mathbf{I}}|\mathbf{q}_{\mathbf{J}}) H^{(g-h)}_{k-i,l-j}(r,q;\mathbf{p}_{\mathbf{K}/\mathbf{I}}|\mathbf{q}_{\mathbf{L}/\mathbf{J}}) \\
&\quad + H^{(g-1)}_{k+1,l}(r,q;r,\mathbf{p}_{\mathbf{K}}|\mathbf{q}_{\mathbf{L}}) \Big),
\end{aligned} \tag{2-31}
$$

where the last equality holds thanks to the loop equation Eq. (2-19). Therefore:

$$
\begin{aligned}
(y(q)-y(p))H^{(g)}_{k,l}(p,q;\mathbf{p}_{\mathbf{K}}|\mathbf{q}_{\mathbf{L}}) &= \operatorname*{Res}_{r\to p,\tilde{q}^j,\mathbf{p}_{\mathbf{K}}} \frac{\tilde{U}^{(0)}_{0,0}(x(p),q)}{(x(p)-x(r))\,\tilde{U}^{(0)}_{0,0}(x(r),q)} \Big(\\
&\quad \sum_h \sum_{I,J} W^{(h)}_{i+1,j}(r,\mathbf{p}_{\mathbf{I}}|\mathbf{q}_{\mathbf{J}}) H^{(g-h)}_{k-i,l-j}(r,q;\mathbf{p}_{\mathbf{K}/\mathbf{I}}|\mathbf{q}_{\mathbf{L}/\mathbf{J}}) \\
&\quad + H^{(g-1)}_{k+1,l}(r,q;r,\mathbf{p}_{\mathbf{K}}|\mathbf{q}_{\mathbf{L}}) \Big).
\end{aligned} \tag{2-32}
$$

If we divide by $\tilde{U}^{(0)}_{0,0}(x(p),q)$ we obtain:

$$
\begin{aligned}
-h^{(g)}_{k,l}(p,q;\mathbf{p}_{\mathbf{K}}|\mathbf{q}_{\mathbf{L}}) &= \operatorname*{Res}_{r\to p,\tilde{q}^j,p_K} \frac{1}{(x(p)-x(r))\,(y(r)-y(q))} \Big(\\
&\quad \sum_h \sum_{I,J} W^{(h)}_{i+1,j}(r,\mathbf{p}_{\mathbf{I}}|\mathbf{q}_{\mathbf{J}}) h^{(g-h)}_{k-i,l-j}(r,q;\mathbf{p}_{\mathbf{K}/\mathbf{I}}|\mathbf{q}_{\mathbf{L}/\mathbf{J}}) \\
&\quad + h^{(g-1)}_{k+1,l}(r,q;r,\mathbf{p}_{\mathbf{K}}|\mathbf{q}_{\mathbf{L}}) \Big).
\end{aligned} \tag{2-33}
$$

The other half of the theorem is obtained from the fact that for large x:

$$
\operatorname{tr} \frac{1}{x-M_1}\frac{1}{y-M_2} \to \frac{1}{x}\operatorname{tr}\frac{1}{y-M_2} \tag{2-34}
$$

and thus:

$$
H^{(g)}_{k,l}(p,q;\mathbf{p}_{\mathbf{K}}|\mathbf{q}_{\mathbf{L}}) \to \frac{1}{x(p)} \frac{W^{(g)}_{k,l+1}(\mathbf{p}_{\mathbf{K}}|\mathbf{q}_{\mathbf{L}},q)}{dy(q)} \tag{2-35}
$$

when $p \to \infty_x$[5]. $\square$

2.3 Examples, first few terms

Let us solve the recursive definition and give explicit formulae for the simplest functions.

Example $W^{(0)}_{1,1}$:

In particular, definitions Eq. (2-22) and Eq. (2-23) give:

$$
W^{(0)}_{1,1}(p_1|q) = \operatorname*{Res}_{r\to\tilde{q}^j,p_1} \frac{dy(q)\,B(r,p_1)}{(y(r)-y(q))}
$$

[5] ∞_x is the only point on the curve where the meromorphic function x has a simple pole (see [15] for further details).

$$
\begin{aligned}
&= -\operatorname*{Res}_{r\to q} \frac{dy(q)\,B(r,p_1)}{(y(r)-y(q))} \\
&= -B(q,p_1).
\end{aligned}
\tag{2-36}
$$

Therefore we recover:

$$
W^{(0)}_{2,0}(p_1,q) + W^{(0)}_{1,1}(p_1|q) = 0. \tag{2-37}
$$

Example $H^{(0)}_{1,0}$:

$$
\begin{aligned}
h^{(0)}_{1,0}(p,q;p_1) &= \operatorname*{Res}_{r\to \tilde{q}^j,p,p_1} \frac{B(r,p_1)}{(x(p)-x(r))(y(r)-y(q))} \\
&= -\operatorname*{Res}_{r\to p^i,q} \frac{B(r,p_1)}{(x(p)-x(r))(y(r)-y(q))}.
\end{aligned}
\tag{2-38}
$$

Example $H^{(0)}_{0,1}$:

$$
\begin{aligned}
h^{(0)}_{0,1}(p,q;p_1) &= \operatorname*{Res}_{r\to \tilde{q}^j,p} \frac{W^{(0)}_{1,1}(r|p_1)}{(x(p)-x(r))(y(r)-y(q))} \\
&= -\operatorname*{Res}_{r\to \tilde{q}^j,p} \frac{B(r,p_1)}{(x(p)-x(r))(y(r)-y(q))} \\
&= \operatorname*{Res}_{r\to p^i,q,p_1} \frac{B(r,p_1)}{(x(p)-x(r))(y(r)-y(q))}.
\end{aligned}
\tag{2-39}
$$

Moreover we have:

$$
\begin{aligned}
h^{(0)}_{1,0}(p,q;p_1) + h^{(0)}_{0,1}(p,q;p_1) &= \operatorname*{Res}_{r\to p_1} \frac{B(r,p_1)}{(x(p)-x(r))(y(r)-y(q))} \\
&= d_{p_1}\left(\frac{1}{(x(p)-x(p_1))(y(p_1)-y(q))}\right).
\end{aligned}
\tag{2-40}
$$

Example $W^{(0)}_{2,1}$:

$$
\begin{aligned}
\frac{W^{(0)}_{2,1}(p_1,p_2|q)}{dy(q)} &= \operatorname{Res}_{r\to \tilde{q}^j,p_1,p_2} \frac{B(r,p_1)h^{(0)}_{1,0}(r,q;p_2)+B(r,p_2)h^{(0)}_{1,0}(r,q;p_1)+W^{(0)}_{3,0}(r,p_1,p_2)}{(y(r)-y(q))} \\
&= -\operatorname{Res}_{r\to q,\mathbf{a}} \frac{B(r,p_1)h^{(0)}_{1,0}(r,q;p_2)+B(r,p_2)h^{(0)}_{1,0}(r,q;p_1)+W^{(0)}_{3,0}(r,p_1,p_2)}{(y(r)-y(q))}.
\end{aligned}
\tag{2-41}
$$

2.4 Conclusion of section 2

Therefore, through theorem 2.1, we have an effective explicit method to compute any $H^{(g)}_{k,l}$ and any $W^{(g)}_{k,l}$ for the 2-matrix model.

This is an interesting result in itself, since none of those quantities were computed before, and those quantities are of importance in applications of random matrices to combinatorics of maps with colored boundaries, i.e. boundary conformal field theory.

An important remark, is that we have chosen to emphasize the role of the loop equation 2-19, rather than equation 2-18, i.e. we have used the Lagrange interpolation formula for a polynomial in x, whereas we could have done the same thing with a polynomial in y. In other words, we have chosen the x-representation rather than the y-representation, although both methods **must** give the same answer. In particular, given $W_{k,0}$, theorem 2.1 allows to compute $W_{0,l}$. $W_{k,0}$ can be computed with the method of [8, 21] using the x-representation, while $W_{0,l}$ can be computed with the method of [8, 21] using the y-representation, i.e. under the exchange

$$x \leftrightarrow y \quad . \tag{2-42}$$

Therefore, in the following section, we improve the result of theorem 2.1, in order to prove that the diagrammatic rules of [8, 21] are indeed symmetric under the exchange of x and y. In other words we prove theorem 7.1 of [21], as announced in that article.

3 Proof of the symmetry x-y of the algebraic invariants $F^{(g)}(\mathcal{E})$

Consider the two algebraic curves:

$$\hat{\mathcal{E}}(x,y) = \mathcal{E}(x,y) \qquad \text{and} \qquad \check{\mathcal{E}}(x,y) = \mathcal{E}(y,x) \tag{3-1}$$

In [21], for any curve $\mathcal{E}$ an infinite sequence of invariants $F^{(g)}$ was defined. Here we consider those invariants for the 2 curves $\hat{\mathcal{E}}$ and $\check{\mathcal{E}}$.

In this section we prove the following theorem (which was announced in [21]):

Theorem 3.1 *Symmetry under the exchange $x \leftrightarrow y$:*

$$\boxed{F^{(g)}(\hat{\mathcal{E}}) = F^{(g)}(\check{\mathcal{E}})} \tag{3-2}$$

where the functional $F^{(g)}(\mathcal{E})$ is defined for any curve $\mathcal{E}(x,y)$ in [21].

3.1 Preliminaries

For the curve $\hat{\mathcal{E}}(x,y) = 0$, we have defined in [21] an infinite sequence of meromorphic forms:

$$\hat{W}_k^{(g)}(p_1,\dots,p_k) = W_k^{(g)}(p_1,\dots,p_k)\Big|_{\hat{\mathcal{E}}} \tag{3-3}$$

with poles only at the zeroes $\mathbf{a} = \{a_i\}$ of dx, and some free energies

$$\hat{F}^{(g)} = \frac{1}{2-2g} \mathop{\mathrm{Res}}_{p\to\mathbf{a}} \Phi(p)\hat{W}_1^{(g)}(p) \tag{3-4}$$

where Φ is any antiderivative of ydx, $d\Phi = ydx$ and $\mathop{\mathrm{Res}}_{p\to\mathbf{a}}$ stands for $\sum_i \mathop{\mathrm{Res}}_{p\to a_i}$.

And likewise, for the curve $\check{\mathcal{E}}(x,y) = 0$, we have defined an infinite sequence of meromorphic forms:

$$\check{W}_k^{(g)}(q_1,\dots,q_k) = W_k^{(g)}(q_1,\dots,q_k)\Big|_{\check{\mathcal{E}}} \tag{3-5}$$

with poles only at the zeroes $\mathbf{b} = \{b_i\}$ of dy, and some free energies

$$\check{F}^{(g)} = \frac{1}{2-2g} \mathop{\mathrm{Res}}_{q\to\mathbf{b}} \Psi(q)\check{W}_1^{(g)}(q) \tag{3-6}$$

where $d\Psi = xdy$.

Our first step is to extend those forms into two families of multilinear meromorphic forms similar to those of section 2 (i.e. mimicking the mixed traces of matrix models):

$$\hat{W}_{k,l}^{(g)}(p_1,\dots,p_k|q_1,\dots,q_l) \qquad \text{and} \qquad \check{W}_{k,l}^{(g)}(p_1,\dots,p_k|q_1,\dots,q_l) \tag{3-7}$$

such that:

$$\hat{W}_{k,0}^{(g)} = \hat{W}_k^{(g)} \qquad , \qquad \check{W}_{0,l}^{(g)} = \check{W}_l^{(g)}. \tag{3-8}$$

Our second step, is to prove that:

$$\hat{W}_{k,l}^{(g)} = \check{W}_{k,l}^{(g)}. \tag{3-9}$$

Our third step, is to prove that:

$$\hat{W}_{k+1,l}^{(g)}(\mathbf{p_K},p|\mathbf{q_L}) + \check{W}_{k,l+1}^{(g)}(\mathbf{p_K}|p,\mathbf{q_L}) = d_p\left(\frac{A_{k,l}^{(g)}(p;\mathbf{p_K}|\mathbf{q_L})}{dx(p)dy(p)}\right) \tag{3-10}$$

where $A_{k,l}^{(g)}(p;\mathbf{p_K}|\mathbf{q_L})$ has poles of degree at most 2 at the poles of ydx, so that in particular for $k=l=0$ we have:

$$\hat{W}_{1,0}^{(g)}(p) + \check{W}_{0,1}^{(g)}(p) = d_p\left(\frac{A_{0,0}^{(g)}(p)}{dx(p)dy(p)}\right) \tag{3-11}$$

where $A_{0,0}^{(g)}$ has poles of degree at most 2 at the poles of ydx.

This last step is sufficent to prove that

$$\hat{F}^{(g)} = \check{F}^{(g)}. \tag{3-12}$$

3.2 Definitions of mixed correlators $\hat{W}_{k,l}^{(g)}$ and $\check{W}_{k,l}^{(g)}$

We define the initial terms:

$$\hat{E}_{0,0}^{(0)}(x,y) = \check{E}_{0,0}^{(0)}(x,y) = \mathcal{E}(x,y), \tag{3-13}$$

$$\hat{H}_{0,0}^{(0)}(p,q) = \check{H}_{0,0}^{(0)}(p,q) = \frac{\mathcal{E}(x(p),y(q))}{(x(p)-x(q))(y(p)-y(q))}, \tag{3-14}$$

$$\hat{W}_{1,0}^{(0)}(p) = \hat{W}_{0,1}^{(0)}(p) = \check{W}_{1,0}^{(0)}(p) = \check{W}_{0,1}^{(0)}(p) = 0, \tag{3-15}$$

$$\hat{W}_{2,0}^{(0)}(p,q) = \hat{W}_{0,2}^{(0)}(p,q) = -\hat{W}_{1,1}^{(0)}(p,q) = B(p,q), \tag{3-16}$$

and

$$\check{W}_{2,0}^{(0)}(p,q) = \check{W}_{0,2}^{(0)}(p,q) = -\check{W}_{1,1}^{(0)}(p,q) = B(p,q). \tag{3-17}$$

Let us define recursively the following quantities for any $g,k,l \geq 0$:

$$\begin{aligned}
&J_{k,l}^{(g)}(p,q;\mathbf{p_K}|\mathbf{q_L}) := \\
&\sum_{m_1,m_2=0}^{k}\sum_{n_1,n_2=0}^{l}\sum_{h,h'=1}^{g}\hat{W}_{m_1+1,n_1}^{(h)}(p,\mathbf{p_{M_1}}|\mathbf{q_{N_1}})\times \\
&\qquad\times\check{W}_{m_2,n_2+1}^{(h')}(\mathbf{p_{M_2}}|\mathbf{q_{N_2}},q)H_{k-m_1-m_2,l-n_1-n_2}^{(g-h-h')}(p,q;\mathbf{p_{K/\{M_1\bigcup M_2\}}}|\mathbf{q_{L/\{N_1\bigcup N_2\}}}) \\
&+\sum_{h=1}^{g-1}\left[(x(p)-x(q))\hat{W}_{1,0}^{(h)}(p)dy(q)+(y(q)-y(p))\check{W}_{0,1}^{(h)}(q)dx(p)\right]H_{k,l}^{(g-h)}(p,q;\mathbf{p_K}|\mathbf{q_L}) \\
&+\sum_{m=0}^{k}\sum_{n=0;mn\neq kl}^{l}\sum_{h=0}^{g}H_{k-m,l-n}^{(g-h)}(p,q;\mathbf{p_{K/M}}|\mathbf{q_{L/N}})\times \\
&\quad\times\left[(x(p)-x(q))\hat{W}_{m+1,n}^{(h)}(p,\mathbf{p_M}|\mathbf{q_N})dy(q)+(y(q)-y(p))\check{W}_{m,n+1}^{(h)}(\mathbf{p_M}|\mathbf{q_N},q)dx(p)\right] \\
&+(x(p)-x(q))H_{k+1,l}^{(g-1)}(p,q;p,\mathbf{p_K}|\mathbf{q_L})dy(q)+(y(q)-y(p))H_{k,l+1}^{(g-1)}(p,q;\mathbf{p_K}|\mathbf{q_L},q)dx(p) \\
&+\sum_{m=0}^{k}\sum_{n=0}^{l}\sum_{h=0}^{g-1}\Big[\hat{W}_{m+1,n}^{(h)}(p,\mathbf{p_M}|\mathbf{q_N})H_{k-m,l-n+1}^{(g-h-1)}(p,q;\mathbf{p_{K/M}}|\mathbf{q_{L/N}},q) \\
&+\tfrac{1}{2}\left(\hat{W}_{m+1,n+1}^{(h)}(p,\mathbf{p_M}|\mathbf{q_N},q)+\check{W}_{m+1,n+1}^{(h)}(p,\mathbf{p_M}|\mathbf{q_N},q)\right)H_{k-m,l-n}^{(g-h-1)}(p,q;\mathbf{p_{K/M}}|\mathbf{q_{L/N}}) \\
&+\check{W}_{m,n+1}^{(h)}(\mathbf{p_M}|\mathbf{q_N},q)H_{k-m+1,l-n}^{(g-h-1)}(p,q;p,\mathbf{p_{K/M}}|\mathbf{q_{L/N}})\Big]+H_{k+1,l+1}^{(g-2)}(p,q;p,\mathbf{p_K}|\mathbf{q_L},q)
\end{aligned} \tag{3-18}$$

and

$$
\begin{aligned}
&\mathcal{J}_{k,l}^{(g)}(p,q;\mathbf{p_K}|\mathbf{q_L}) := J_{k,l}^{(g)}(p,q;\mathbf{p_K}|\mathbf{q_L}) \\
&-\sum_{\alpha=1}^{k} d_{p_\alpha} \Bigg\{ \tfrac{dx(p)}{x(p)-x(p_\alpha)} \Bigg[(x(p_\alpha)-x(q))dy(q)\, H_{k-1,l}^{(g)}(p_\alpha,q;\mathbf{p_{K-\{\alpha\}}}|\mathbf{q_L}) \\
&+\sum_{h=1}^{g} \check{W}_{0,1}^{(h)}(q) H_{k-1,l}^{(g-h)}(p_\alpha,q;\mathbf{p_{K-\{\alpha\}}}|\mathbf{q_L}) \\
&+\sum_{h}\sum_{i,j=0}^{<kl} H_{i-1,j}^{(g-h)}(p_\alpha,q;\mathbf{p_{I-\{\alpha\}}}|\mathbf{q_J}) \check{W}_{k-i,l-j+1}^{(h)}(\mathbf{p_{K-I}}|\mathbf{q_{L-J}},q) \Bigg] \Bigg\} \\
&-\sum_{\beta=1}^{l} d_{q_\beta} \Bigg\{ \tfrac{dy(q)}{y(q)-y(q_\beta)} \Bigg[(y(q_\beta)-y(p))dx(p)\, H_{k,l-1}(p,q_\beta;\mathbf{p_K}|\mathbf{q_{L-\{\beta\}}}) \\
&+\sum_{h=1}^{g} \hat{W}_{1,0}^{(h)}(p) H_{k,l-1}^{(g-h)}(p,q_\beta;\mathbf{p_K}|\mathbf{q_{L-\{\beta\}}}) \\
&+\sum_{i,j=0}^{<kl}\sum_{h} H_{i,j-1}^{(g-h)}(p,q_\beta;\mathbf{p_I}|\mathbf{q_{J-\{\beta\}}}) \hat{W}_{k-i+1,l-j}^{h}(p,\mathbf{p_{K-I}}|\mathbf{q_{L-J}}) \Bigg] \Bigg\} \\
&+\sum_{\alpha=1}^{k}\sum_{\beta=1}^{l} d_{p_\alpha} d_{q_\beta} \left\{ \tfrac{dx(p)}{x(p)-x(p_\alpha)} \tfrac{dy(q)}{y(q)-y(q_\beta)} H_{k-1,l-1}(p_\alpha,q_\beta;\mathbf{p_{K-\{\alpha\}}}|\mathbf{q_{L-\{\beta\}}}) \right\} \\
&-\sum_{\alpha=1}^{k} d_{p_\alpha} \left(\tfrac{H_{k-1,l+1}^{(g-1)}(p_\alpha,q;\mathbf{p_{K-\{\alpha\}}}|\mathbf{q_L},q)dx(p)}{x(p)-x(p_\alpha)} \right) - \sum_{\beta=1}^{l} d_{q_\beta} \left(\tfrac{H_{k+1,l-1}^{(g-1)}(p,q_\beta;p,\mathbf{p_K}|\mathbf{q_{L-\{\beta\}}})dy(q)}{y(q)-y(q_\beta)} \right).
\end{aligned}
\tag{3-19}
$$

Remark 3.1 Those expressions are not as complicated as they look. They are inspired from section 2. In the matrix model case of section 2, those expressions contain nearly all the terms we would obtain from inserting loop equation 2-19 into loop equation 2-20, or equivalently, from inserting loop equation 2-18 into loop equation 2-21. However, here we are not in a matrix model, and we don't assume any of the equations 2-19 to 2-21, in fact we are going to prove them.

Now we define:

$$
\begin{aligned}
&\hat{W}_{k+1,l}^{(g)}(p,\mathbf{p_K}|\mathbf{q_L}) \\
:= \quad &\operatorname*{Res}_{s\to\mathbf{a},\mathbf{q_L}} dS_{s,o}(p) \left[\frac{1}{d_1}\sum_{j=1}^{d_1} \frac{\mathcal{J}_{k,l}^{(g)}(s,\tilde{s}^j;\mathbf{p_K}|\mathbf{q_L})}{U_{0,0}^{(0)}(s,y(s))dy(s)} + \frac{1}{d_2}\sum_{i=1}^{d_2} \frac{\mathcal{J}_{k,l}^{(g)}(s^i,s;\mathbf{p_K}|\mathbf{q_L})}{\widetilde{U}_{0,0}^{(0)}(x(s),s)dx(s)} \right],
\end{aligned}
\tag{3-20}
$$

$$
\begin{aligned}
&\hat{W}_{k,l+1}^{(g)}(\mathbf{p_K}|\mathbf{q_L},q) \\
:= \quad &\operatorname*{Res}_{s\to\mathbf{b},\mathbf{p_K}} dS_{s,o}(q) \left[\frac{1}{d_1}\sum_{j=1}^{d_1} \frac{\mathcal{J}_{k,l}^{(g)}(s,\tilde{s}^j;\mathbf{p_K}|\mathbf{q_L})}{U_{0,0}^{(0)}(s,y(s))dy(s)} + \frac{1}{d_2}\sum_{i=1}^{d_2} \frac{\mathcal{J}_{k,l}^{(g)}(s^i,s;\mathbf{p_K}|\mathbf{q_L})}{\widetilde{U}_{0,0}^{(0)}(x(s),s)dx(s)} \right],
\end{aligned}
\tag{3-21}
$$

$$
\begin{aligned}
&G^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L}) \\
&:= J^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L}) + H^{(0)}_{0,0}(p,q) \\
&\left[(x(p)-x(q))\hat{W}^{(g)}_{k+1,l}(p,\mathbf{p_K}|\mathbf{q_L})dy(q) + (y(q)-y(p))\check{W}^{(g)}_{k,l+1}(\mathbf{p_K}|\mathbf{q_L},q)dx(p)\right],
\end{aligned}
\tag{3-22}
$$

$$
\begin{aligned}
&\mathcal{G}^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L}) \\
&:= \mathcal{J}^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L}) + H^{(0)}_{0,0}(p,q) \\
&\left[(x(p)-x(q))\hat{W}^{(g)}_{k+1,l}(p,\mathbf{p_K}|\mathbf{q_L})dy(q) + (y(q)-y(p))\check{W}^{(g)}_{k,l+1}(\mathbf{p_K}|\mathbf{q_L},q)dx(p)\right],
\end{aligned}
\tag{3-23}
$$

$$
\frac{\widehat{H}^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L})}{\mathcal{E}(x(p),y(q))} := \mathop{\mathrm{Res}}_{r\to q,p^i} \frac{\mathcal{G}^{(g)}_{k,l}(p,r;\mathbf{p_K}|\mathbf{q_L})}{(y(q)-y(p))(y(q)-y(r))(x(p)-x(r))H^{(0)}_{0,0}(p,r)dx(p)},
\tag{3-24}
$$

$$
\frac{\check{H}^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L})}{\mathcal{E}(x(p),y(q))} := \mathop{\mathrm{Res}}_{r\to p,\tilde{q}^j} \frac{\mathcal{G}^{(g)}_{k,l}(r,q;\mathbf{p_K}|\mathbf{q_L})}{(x(p)-x(q))(x(p)-x(r))(y(q)-y(r))H^{(0)}_{0,0}(r,q)dy(q)},
\tag{3-25}
$$

and

$$
H^{(g)}_{k,l} = \frac{\hat{H}^{(g)}_{k,l} + \check{H}^{(g)}_{k,l}}{2}
\tag{3-26}
$$

(we prove below that $\hat{H}^{(g)}_{k,l} = \check{H}^{(g)}_{k,l} = H^{(g)}_{k,l}$) as well as

$$
\frac{\widehat{E}^{(g)}_{k,l}(p,q,\mathbf{p_K}|\mathbf{q_L})}{\mathcal{E}(x(p),y(q))} := \mathop{\mathrm{Res}}_{r\to p^i} \frac{\mathcal{G}^{(g)}_{k,l}(p,r;\mathbf{p_K}|\mathbf{q_L})}{(y(q)-y(p))(y(q)-y(r))(x(p)-x(r))H^{(0)}_{0,0}(p,r)dx(p)},
\tag{3-27}
$$

$$
\frac{\check{E}^{(g)}_{k,l}(p,q,\mathbf{p_K}|\mathbf{q_L})}{\mathcal{E}(x(p),y(q))} := \mathop{\mathrm{Res}}_{r\to \tilde{q}^j} \frac{\mathcal{G}^{(g)}_{k,l}(r,q;\mathbf{p_K}|\mathbf{q_L})}{(x(p)-x(q))(x(p)-x(r))(y(q)-y(r))H^{(0)}_{0,0}(r,q)dy(q)},
\tag{3-28}
$$

$$
\begin{aligned}
-\tilde{U}^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L}) \quad := \quad & (y(q)-y(p))H^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L}) \\
&+\sum_h\sum_{I,J}\frac{\hat{W}^{(h)}_{i+1,j}(p,\mathbf{p_I}|\mathbf{q_J})H^{(g-h)}_{k-i,l-j}(p,q;\mathbf{p_{K/I}}|\mathbf{q_{L/J}})}{dx(p)} \\
&+\frac{H^{(g-1)}_{k+1,l}(p,q;p,\mathbf{p_K}|\mathbf{q_L})}{dx(p)^2} \\
&-\sum_m d_{p_m}\frac{H^{(g)}_{k-1,l}(p_m,q;\mathbf{p_{K/\{m\}}}|\mathbf{q_L})}{x(p)-x(p_m)}
\end{aligned}
$$

(3 − 29)

and

$$
\begin{aligned}
-U^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L}) \quad := \quad & (x(p)-x(q))H^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L}) \\
& +\sum_h \sum_{I,J} \frac{\check{W}^{(h)}_{i,j+1}(\mathbf{p_I}|\mathbf{q_J},q)H^{(g-h)}_{k-i,l-j}(p,q;\mathbf{p_{K/I}}|\mathbf{q_{L/J}})}{dy(q)} \\
& +\frac{H^{(g-1)}_{k,l+1}(p,q;\mathbf{p_K}|\mathbf{q_L},q)}{dy(q)^2} \\
& -\sum_n d_{q_n} \frac{H^{(g)}_{k,l-1}(p,q_n;\mathbf{p_K}|\mathbf{q_{L/\{n\}}})}{y(q)-y(q_n)}.
\end{aligned}
\tag{3-30}
$$

Those definitions form a triangular system of definitions, and each term is well defined in a unique recursive way.

Remark 3.2 Definitions eq.3-29 and eq.3-30 coincide with loop equation 2-19 and 2-18 in the matrix model case, i.e. when $\mathcal{E}$ is the classical spectral curve of the 2 matrix model.

3.3 Theorems

Theorem 3.2 *For $2g+k+l \geq 3$, one has the following properties:*

- $\hat{W}^{(g)}_{k,l}(\mathbf{p_K}|\mathbf{q_L})$ *(resp. $\check{W}^{(g)}_{k,l}(\mathbf{p_K}|\mathbf{q_L})$) has poles only when $p_i \to \mathbf{a},\mathbf{q_L}$ and $q_j \to \mathbf{b},\mathbf{p_K}$;*
- *in any of the $k+l$ variables, the $\mathcal{A}$-cycle integrals vanish: $\oint_{\mathcal{A}} \hat{W}^{(g)}_{k,l} = \oint_{\mathcal{A}} \check{W}^{(g)}_{k,l} = 0$;*
- $\widehat{H}^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L}) = \check{H}^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L})$ *has poles only when $p \to q,\mathbf{a},\mathbf{q_L}$ and $q \to p,\mathbf{b},\mathbf{p_K}$, and*

$$\widehat{E}^{(g)}_{k,l}(x(p),q;\mathbf{p_K}|\mathbf{q_L}) = \check{E}^{(g)}_{k,l}(p,y(q);\mathbf{p_K}|\mathbf{q_L}) := E^{(g)}(x(p),y(q);\mathbf{p_K}|\mathbf{q_L}) \tag{3-31}$$

  is a polynomial of degree d_1-1 in $x(p)$ and d_2-1 in $y(q)$;
- $U^{(g)}_{k,l}(p,y(q);\mathbf{p_K}|\mathbf{q_L})$ *(resp. $\widetilde{U}^{(g)}_{k,l}(x(p),q;\mathbf{p_K};\mathbf{q_L})$) is a polynomial in $y(q)$ (resp. $x(p)$) of degree d_2-1 (resp. d_1-1).*

proof:

Let us proceed by induction on $2g+k+l$. Suppose that the properties are satisfied for any g',k',l' such that $2g'+k'+l' < 2g+k+l$. Let us prove that they are true for g,k,l. In order to make the proof more readable, we split it into pieces. Nevertheless, **for every step, the global recursion hypothesis is needed**.

We need the following lemma:

Lemma 3.1 *The quantity*

$$f_{k,l}^{(g)}(s;\mathbf{p_K};\mathbf{q_L}) := \frac{\mathcal{J}_{k,l}^{(g)}(s,\tilde{s}^j;\mathbf{p_K};\mathbf{q_L})}{U_{0,0}^{(0)}(s,y(s))dy(s)} \tag{3-32}$$

is independent of $j \neq 0$, it is a meromorphic one-form in the variable s, with poles at $s = \mathbf{a}, \mathbf{q_L}$, and it vanishes to order at least $\deg(ydx) - 1$ near the poles of ydx.

Similarly, the quantity

$$\tilde{f}_{k,l}^{(g)}(s;\mathbf{p_K};\mathbf{q_L}) := \frac{\mathcal{J}_{k,l}^{(g)}(s^i,s;\mathbf{p_K};\mathbf{q_L})}{\widetilde{U}_{0,0}^{(0)}(x(s),s)dx(s)}. \tag{3-33}$$

is independent of $i \neq 0$, it is a meromorphic one-form in the variable s, with poles at $s = \mathbf{b}, \mathbf{q_L}$, and it vanishes to order at least $\deg(xdy) - 1$ near the poles of xdy.

Moreover one has:

$$\oint_{\mathcal{A}} \left(f_{k,l}^{(g)}(s;\mathbf{p_K};\mathbf{q_L}) + \tilde{f}_{k,l}^{(g)}(s;\mathbf{p_K};\mathbf{q_L}) \right) = 0, \tag{3-34}$$

$$\oint_{\mathcal{B}} \left(f_{k,l}^{(g)}(s;\mathbf{p_K};\mathbf{q_L}) + \tilde{f}_{k,l}^{(g)}(s;\mathbf{p_K};\mathbf{q_L}) \right) = 0 \tag{3-35}$$

and:

$$\begin{aligned} & f_{k,l}^{(g)}(s;\mathbf{p_K};\mathbf{q_L}) + \tilde{f}_{k,l}^{(g)}(s;\mathbf{p_K};\mathbf{q_L}) \\ = & \operatorname*{Res}_{q\to\mathbf{a},\mathbf{b},\mathbf{p_K},\mathbf{q_L}} dS_{q,o}(s) \Big(f_{k,l}^{(g)}(q;\mathbf{p_K};\mathbf{q_L}) + \tilde{f}_{k,l}^{(g)}(q;\mathbf{p_K};\mathbf{q_L}) \Big). \end{aligned} \tag{3-36}$$

Proof of the lemma:

First of all, One can remark that the definition of $\mathcal{J}_{k,l}^{(g)}$ involves only quantities whose properties are known by the recursion hypothesis. One can note that it can be written under the following forms:

$$
\begin{aligned}
& J^{(g)}_{k,l}(p,q;\mathbf{p_K};\mathbf{q_L}) \\
& := -\sum_{h=1}^{g-1}\sum_{m=0}^{k}\sum_{n=0}^{l}\check{W}^{(h)}_{m,n+1}(\mathbf{p_M};q,\mathbf{q_N})\widetilde{U}^{g-h}_{k-m,l-n}(x(p),q;\mathbf{p_{K/M}},\mathbf{q_{L/N}})dx(p) \\
& -\sum_{m=0}^{k}\sum_{n=0,(m,n)\neq(0,0)}^{l}\check{W}^{(0)}_{m,n+1}(\mathbf{p_M};q,\mathbf{q_N})\widetilde{U}^{g}_{k-m,l-n}(x(p),q;\mathbf{p_{K/M}},\mathbf{q_{L/N}})dx(p) \\
& -\sum_{m=0}^{k}\sum_{n=0,(m,n)\neq(k,l)}^{l}\check{W}^{(g)}_{m,n+1}(\mathbf{p_M};q,\mathbf{q_N})\widetilde{U}^{0}_{k-m,l-n}(x(p),q;\mathbf{p_{K/M}},\mathbf{q_{L/N}})dx(p) \\
& -\widetilde{U}^{(g-1)}_{k,l+1}(x(p),q;\mathbf{p_K};q,\mathbf{q_L})dx(p) \\
& +\sum_{h=1}^{g-1}\sum_{m=0}^{k}\sum_{n=0}^{l}d_{p_\alpha}\left(\frac{\check{W}^{(h)}_{m,n+1}(\mathbf{p_M};q,\mathbf{q_N})H^{g-h}_{k-m-1,l-n}(p_\alpha,q;\mathbf{p_{K/M/\{\alpha\}}},\mathbf{q_{L/N}})dx(p)}{x(p)-x(p_\alpha)}\right) \\
& +\sum_{m=0}^{k}\sum_{n=0,(m,n)\neq(0,0)}^{l}d_{p_\alpha}\left(\frac{\check{W}^{(0)}_{m,n+1}(\mathbf{p_M};q,\mathbf{q_N})H^{g}_{k-m-1,l-n}(p_\alpha,q;\mathbf{p_{K/M/\{\alpha\}}},\mathbf{q_{L/N}})dx(p)}{x(p)-x(p_\alpha)}\right) \\
& +\sum_{m=0}^{k}\sum_{n=0,(m,n)\neq(k,l)}^{l}d_{p_\alpha}\left(\frac{\check{W}^{(g)}_{m,n+1}(\mathbf{p_M};q,\mathbf{q_N})H^{0}_{k-m-1,l-n}(p_\alpha,q;\mathbf{p_{K/M/\{\alpha\}}},\mathbf{q_{L/N}})dx(p)}{x(p)-x(p_\alpha)}\right) \\
& +d_{p_\alpha}\left(\frac{H^{(g-1)}_{k-1,l+1}(p_\alpha,q;\mathbf{p_{K/\{\alpha\}}};q,\mathbf{q_L})dx(p)}{x(p)-x(p_\alpha)}\right) \\
& +(x(p)-x(q))\left[\sum_{h=1}^{g-1}\sum_{m=0}^{k}\sum_{n=0}^{l}\hat{W}^{(h)}_{m+1,n}(p,\mathbf{p_M};\mathbf{q_N})dy(q)H^{(g-h)}_{k-m,l-n}(p,q;\mathbf{p_{K/M}};\mathbf{q_{L/N}})\right. \\
& +\textstyle\sum_{m=0}^{k}\sum_{n=0,(m,n)\neq(k,l)}^{l}\hat{W}^{(0)}_{m+1,n}(p,\mathbf{p_M};\mathbf{q_N})dy(q)H^{(g)}_{k-m,l-n}(p,q;\mathbf{p_{K/M}};\mathbf{q_{L/N}}) \\
& +\textstyle\sum_{m=0}^{k}\sum_{n=0,(m,n)\neq(0,0)}^{l}\hat{W}^{(g)}_{m+1,n}(p,\mathbf{p_M};\mathbf{q_N})dy(q)H^{(0)}_{k-m,l-n}(p,q;\mathbf{p_{K/M}};\mathbf{q_{L/N}}) \\
& \left.+H^{(g-1)}_{k+1,l}(p,q;p,\mathbf{p_K};\mathbf{q_L})dy(q)\right]
\end{aligned}
\tag{3-37}
$$

and

$$
\begin{aligned}
&\mathcal{J}^{(g)}_{k,l}(p,q;\mathbf{p_K};\mathbf{q_L})\\
&= -\sum_{h=1}^{g-1}\sum_{m=0}^{k}\sum_{n=0}^{l}\check{W}^{(h)}_{m,n+1}(\mathbf{p_M};q,\mathbf{q_N})\widetilde{U}^{g-h}_{k-m,l-n}(x(p),q;\mathbf{p_{K/m}},\mathbf{q_{L/N}})dx(p)\\
&-\sum_{m=0}^{k}\sum_{n=0,(m,n)\neq(0,0)}^{l}\check{W}^{(0)}_{m,n+1}(\mathbf{p_M};q,\mathbf{q_N})\widetilde{U}^{g}_{k-m,l-n}(x(p),q;\mathbf{p_{K/m}},\mathbf{q_{L/N}})dx(p)\\
&-\sum_{m=0}^{k}\sum_{n=0,(m,n)\neq(k,l)}^{l}\check{W}^{(g)}_{m,n+1}(\mathbf{p_M};q,\mathbf{q_N})\widetilde{U}^{0}_{k-m,l-n}(x(p),q;\mathbf{p_{K/m}},\mathbf{q_{L/N}})dx(p)\\
&-\widetilde{U}^{(g-1)}_{k,l+1}(x(p),q;\mathbf{p_K};q,\mathbf{q_L})dx(p)\\
&+d_{q_\beta}\left(\frac{\widetilde{U}^{(g)}_{k,l-1}(p,q_\beta;\mathbf{p_K};\mathbf{q_{L/\{\beta\}}})}{y(q)-y(q_\beta)}dx(p)dy(q)\right)\\
&-d_{p_\alpha}\left(\frac{x(p_\alpha)-x(q)}{x(p)-x(p_\alpha)}H^{(g)}_{k-1,l}(p_\alpha,q;\mathbf{p_{K:\{\alpha\}}},\mathbf{q_L})dx(p)dy(q)\right)\\
&+(x(p)-x(q))\left[\sum_{h=1}^{g-1}\sum_{m=0}^{k}\sum_{n=0}^{l}\hat{W}^{(h)}_{m+1,n}(p,\mathbf{p_M};\mathbf{q_N})dy(q)H^{(g-h)}_{k-m,l-n}(p,q;\mathbf{p_{K/M}};\mathbf{q_{L/N}})\right.\\
&+\sum_{m=0}^{k}\sum_{n=0,(m,n)\neq(k,l)}^{l}\hat{W}^{(0)}_{m+1,n}(p,\mathbf{p_M};\mathbf{q_N})dy(q)H^{(g)}_{k-m,l-n}(p,q;\mathbf{p_{K/M}};\mathbf{q_{L/N}})\\
&+\sum_{m=0}^{k}\sum_{n=0,(m,n)\neq(0,0)}^{l}\hat{W}^{(g)}_{m+1,n}(p,\mathbf{p_M};\mathbf{q_N})dy(q)H^{(0)}_{k-m,l-n}(p,q;\mathbf{p_{K/M}};\mathbf{q_{L/N}})\\
&\left.+H^{(g-1)}_{k+1,l}(p,q;p,\mathbf{p_K};\mathbf{q_L})\right]dy(q).
\end{aligned}
\tag{3-38}
$$

Thanks to the properties implied by the recursion hypothesis (U and $\tilde{U}$ are polynomials), one has:

$$
\begin{aligned}
&\mathcal{J}^{(g)}_{k,l}(q^i,q;\mathbf{p_K};\mathbf{q_L})\\
=\quad &-\sum_{h=1}^{g-1}\sum_{m=0}^{k}\sum_{n=0}^{l}\check{W}^{(h)}_{m,n+1}(\mathbf{p_M};q,\mathbf{q_N})\widetilde{U}^{g-h}_{k-m,l-n}(x(q),q;\mathbf{p_{K/M}},\mathbf{q_{L/N}})dx(q)\\
&-\sum_{m=0}^{k}\sum_{n=0,(m,n)\neq(0,0)}^{l}\check{W}^{(0)}_{m,n+1}(\mathbf{p_M};q,\mathbf{q_N})\widetilde{U}^{g}_{k-m,l-n}(x(q),q;\mathbf{p_{K/M}},\mathbf{q_{L/N}})dx(q)\\
&-\sum_{m=0}^{k}\sum_{n=0,(m,n)\neq(k,l)}^{l}\check{W}^{(g)}_{m,n+1}(\mathbf{p_M};q,\mathbf{q_N})\widetilde{U}^{0}_{k-m,l-n}(x(q),q;\mathbf{p_{K/M}},\mathbf{q_{L/N}})dx(q)\\
&-\widetilde{U}^{(g-1)}_{k,l+1}(x(q),q;\mathbf{p_K};q,\mathbf{q_L})dx(q)\\
&+d_{q_\beta}\left(\frac{\widetilde{U}^{(g)}_{k,l-1}(x(q),q_\beta;\mathbf{p_K};\mathbf{q_{L/\{\beta\}}})}{y(q)-y(q_\beta)}\right)dx(q)dy(q)\\
&+d_{p_\alpha}\left(H^{(g)}_{k-1,l}(p_\alpha,q;\mathbf{p_{K/\{\alpha\}}},\mathbf{q_L})\right)dx(q)dy(q)
\end{aligned}
$$

$(3-39)$

for any non vanishing i. Thus this quantity does not depend on i, and $\tilde{f}$ is clearly a meromorphic 1-form, whose poles can be easily seen on this expression using the recursion hypothesis.

The same considerations give the equivalent through the exchange of $x \leftrightarrow y$:

$$\begin{aligned} & \mathcal{J}_{k,l}^{(g)}(p,\tilde{p}^j;\mathbf{p_K};\mathbf{q_L}) \\ = \quad & -\sum_{h=1}^{g-1}\sum_{m=0}^{k}\sum_{n=0}^{l} \hat{W}_{m+1,n}^{(h)}(p,\mathbf{p_M};\mathbf{q_N}) U_{k-m,l-n}^{g-h}(p,y(p);\mathbf{p_{K/M}},\mathbf{q_{L/N}})dy(p) \\ & -\sum_{m=0}^{k}\sum_{n=0,(m,n)\neq(0,0)}^{l} \hat{W}_{m+1,n}^{(0)}(p,\mathbf{p_M};\mathbf{q_N}) U_{k-m,l-n}^{g}(p,y(p);\mathbf{p_{K/M}},\mathbf{q_{L/N}})dy(p) \\ & -\sum_{m=0}^{k}\sum_{n=0,(m,n)\neq(k,l)}^{l} \hat{W}_{m+1,n}^{(g)}(p,\mathbf{p_M};\mathbf{q_N}) U_{k-m,l-n}^{0}(p,y(p);\mathbf{p_{K/M}},\mathbf{q_{L/N}})dy(p) \\ & -U_{k+1,l}^{(g-1)}(p,y(p);p,\mathbf{p_K};\mathbf{q_L})dy(p) \\ & +d_{p_\alpha}\left(\frac{U_{k-1,l}^{(g)}(p_\alpha,y(p);\mathbf{p_{K/\{\alpha\}}};\mathbf{q_L})}{x(p)-x(p_\alpha)}\right)dx(p)dy(p) \\ & +d_{q_\beta}\left(H_{k,l-1}^{(g)}(p,q_\beta;\mathbf{p_{K/\{\alpha\}}},\mathbf{q_{L/\{\beta\}}})\right)dx(p)dy(p) \end{aligned} \tag{3-40}$$

This quantity does not depend on j, and f is clearly a meromorphic 1-form, whose poles can be easily seen on this expression using the recursion hypothesis.

The fact that the $\mathcal{A}$ and $\mathcal{B}$ cycle integrals vanish comes from the symmetry $x \leftrightarrow y$. Indeed under the symmetry $x \leftrightarrow y$, f is changed to $\tilde{f}$ and $\tilde{f}$ is changed to f. At the same time the $\mathcal{A}$-cycles are changed to $-\mathcal{A}$ because $2i\pi\epsilon = \oint_{\mathcal{A}} ydx = -\oint_{\mathcal{A}} xdy$, and the $\mathcal{B}$-cycles are changed to $-\mathcal{B}$ in order to form a canonical basis. Therefore, the $\mathcal{A}$ and $\mathcal{B}$ cycle integrals of $f+\tilde{f}$ vanish.

Equation 3-36 simply comes from Cauchy residue formula and Riemann's bilinear identity.

The fact that f vanishes to order at least $\deg(ydx)-1$ near a pole α of ydx follows from the definition of $\mathcal{J}$:

$$\begin{aligned} & \frac{\mathcal{J}_{k,l}^{(g)}(p,\tilde{p}^j;\mathbf{p_K}|\mathbf{q_L})}{dx(p)dy(p)} \sim_{p\to\alpha} \\ \sim_{p\to\alpha} \quad & \frac{x(p)-x(\tilde{p}^j)}{dx(p)}\Big(\sum_{m=0}^{k}\sum_{n=0}^{l}\sum_{h=0}^{g} \hat{W}_{m+1,n}^{(h)}(p,\mathbf{p_M}|\mathbf{q_N}) H_{k-m,l-n}^{(g-h)}(p,\tilde{p}^j;\mathbf{p_{K/M}}|\mathbf{q_{L/N}}) \\ & +H_{k+1,l}^{(g-1)}(p,\tilde{p}^j;p,\mathbf{p_K}|\mathbf{q_L})\Big) \\ & -\sum_{\alpha=1}^{k} d_{p_\alpha}\left(\frac{(x(p_\alpha)-x(\tilde{p}^j))}{x(p)-x(p_\alpha)} H_{k-1,l}^{(g)}(p_\alpha,\tilde{p}^j;\mathbf{p_{K-\{\alpha\}}}|\mathbf{q_L})\right) \end{aligned}$$

$$- \sum_{\beta=1}^{l} d_{q_\beta} \left(\frac{(y(q_\beta) - y(p))}{y(p) - y(q_\beta)} \, H_{k,l-1}(p, q_\beta; \mathbf{p_K}|\mathbf{q}_{\mathbf{L}-\{\beta\}}) \right)$$

(3 − 41)

which is at most finite if p approaches a pole α of ydx. Then it implies that $f^{(g)}_{k,l}(p;\mathbf{p_K};\mathbf{q_L}) = \frac{\mathcal{J}^{(g)}_{k,l}(p,\tilde{p}^j;\mathbf{p_K};\mathbf{q_L})}{U^{(0)}_{0,0}(s,y(s))dy(p)}$ vanishes at order at least $\deg(ydx) - 1$.

The same holds for $\tilde{f}$.

□

• **$W^{(g)}_{k,l}$ has poles only when $p_i \to \mathbf{a}, \mathbf{q_L}$ and $q_j \to \mathbf{b}, \mathbf{p_K}$, and $\oint_{\mathcal{A}} W^{(g)}_{k,l} = 0$.**

From the definition eq.3-20, it is clear that $\hat{W}^{(g)}_{k+1,l}(p, p_1, \ldots, p_k|q_1, \ldots, q_l)$ is finite when p is not close to a branch point or to one of the q_j's, and becomes infinite only if the integration contour is pinched. Thus in the variable p, the only poles of $\hat{W}^{(g)}_{k+1,l}(p, p_1, \ldots, p_k|q_1, \ldots, q_l)$ are at $p = \mathbf{a}, \mathbf{q_L}$.

The poles of $\hat{W}^{(g)}_{k+1,l}(p, p_1, \ldots, p_k|q_1, \ldots, q_l)$ in any other variable, follow from the recursion hypothesis, and thus they are at $p_i = \mathbf{a}, \mathbf{q_L}$, and at $q_j = \mathbf{b}, p, \mathbf{p_K}$.

The fact that $\oint_{\mathcal{A}} \hat{W}^{(g)}_{k+1,l} = 0$ when one integrates over the first variable comes from the fact that this is a property of dS, and in the other variables it comes from the recursion hypothesis.

By a symmetric argument, the same holds for $\check{W}^{(g)}_{k,l+1}(p_1, \ldots, p_k|q_1, \ldots, q_l, p)$, and we see that $\hat{W}^{(g)}_{k,l}$ and $\check{W}^{(g)}_{k,l}$ have the same poles.

We have (from the Cauchy residue formula and Riemann bilinear identity):

$$\hat{W}^{(g)}_{k+1,l}(p, \mathbf{p_K}|\mathbf{q_L}) + \check{W}^{(g)}_{k,l+1}(\mathbf{p_K}|\mathbf{q_L}, p) = f^{(g)}_{k,l}(p; \mathbf{p_K}|\mathbf{q_L}) + \tilde{f}^{(g)}_{k,l}(p; \mathbf{p_K}|\mathbf{q_L}). \tag{3-42}$$

• **$\widehat{H}^{(g)}_{k,l}(p, q; \mathbf{p_K}|\mathbf{q_L}) = \check{H}^{(g)}_{k,l}(p, q; \mathbf{p_K}|\mathbf{q_L})$.**

One has:

$$\begin{aligned}
&\frac{\widehat{H}^{(g)}_{k,l}(p,q;\mathbf{p_K};\mathbf{q_L})}{\mathcal{E}(x(p),y(q))} = \\
&= \operatorname{Res}_{r\to q,p^i} \frac{\mathcal{G}^{(g)}_{k,l}(p,r;\mathbf{p_K};\mathbf{q_L})}{(y(q)-y(p))(y(q)-y(r))(x(p)-x(r))H^{(0)}_{0,0}(p,r)dx(p)} \\
&= \operatorname{Res}_{r\to q,p^i} \operatorname{Res}_{s\to p} \frac{\mathcal{G}^{(g)}_{k,l}(s,r;\mathbf{p_K};\mathbf{q_L})}{(y(q)-y(p))(y(q)-y(r))(x(s)-x(r))(x(s)-x(p))H^{(0)}_{0,0}(s,r)} \\
&= \operatorname{Res}_{r\to q,p^i} \operatorname{Res}_{s\to p,\tilde{q}^j} \frac{\mathcal{G}^{(g)}_{k,l}(s,r;\mathbf{p_K};\mathbf{q_L})}{(y(q)-y(p))(y(q)-y(r))(x(s)-x(r))(x(s)-x(p))H^{(0)}_{0,0}(s,r)}
\end{aligned} \tag{3-43}$$

where the last equality holds because the integrant has no pole when $s \to \tilde{q}^j$. Then

$$\begin{aligned}
&\frac{\widehat{H}^{(g)}_{k,l}(p,q;\mathbf{p_K};\mathbf{q_L})}{\mathcal{E}(x(p),y(q))} = \\
&= \text{Res}_{r\to q,p^i}\ \text{Res}_{s\to p,\tilde{q}^j}\ \frac{\mathcal{G}^{(g)}_{k,l}(s,r;\mathbf{p_K};\mathbf{q_L})}{(y(q)-y(r))(x(s)-x(p))H^{(0)}_{0,0}(s,r)}\Big[\frac{1}{(y(r)-y(s))(x(p)-x(q))} \\
&\quad +\frac{1}{(x(s)-x(r))(y(q)-y(p))} - \frac{1}{(y(r)-y(s))(x(p)-x(q))}\Big] \\
&= \text{Res}_{r\to q,p^i}\ \text{Res}_{s\to p,\tilde{q}^j}\ \frac{\mathcal{G}^{(g)}_{k,l}(s,r;\mathbf{p_K};\mathbf{q_L})}{(y(q)-y(r))(x(s)-x(p))(y(r)-y(s))(x(p)-x(q))H^{(0)}_{0,0}(s,r)} \\
&+\sum_{i=1}^{d_2} \frac{\mathcal{G}^{(g)}_{k,l}(p,p^i;\mathbf{p_K};\mathbf{q_L})}{(y(q)-y(p))(y(q)-y(p^i))H^{(0)}_{0,0}(p,p^i)dx(p)^2}.
\end{aligned} \tag{3-44}$$

Note that the first term corresponds exactly to $\frac{\check{H}^{(g)}_{k,l}(p,q;\mathbf{p_K};\mathbf{q_L})}{\mathcal{E}(x(p),y(q))}$ with the integration contours for r and s exchanged. However, the poles of the integrand are known and thus:

$$\begin{aligned}
\mathop{\text{Res}}_{r\to q,p^i}\mathop{\text{Res}}_{s\to p,\tilde{q}^j} &= \mathop{\text{Res}}_{r\to q}\mathop{\text{Res}}_{s\to p} + \mathop{\text{Res}}_{r\to p^i}\mathop{\text{Res}}_{s\to \tilde{q}^j} + \mathop{\text{Res}}_{r\to q}\mathop{\text{Res}}_{s\to \tilde{q}^j} + \mathop{\text{Res}}_{r\to p^i}\mathop{\text{Res}}_{s\to p} \\
&= \mathop{\text{Res}}_{s\to p}\mathop{\text{Res}}_{r\to q} + \mathop{\text{Res}}_{s\to \tilde{q}^j}\mathop{\text{Res}}_{r\to p^i} + \sum_{j\neq 0}\mathop{\text{Res}}_{\tilde{r}^j\to \tilde{q}^j}\mathop{\text{Res}}_{s\to \tilde{q}^j} + \sum_{i\neq 0}\mathop{\text{Res}}_{r^i\to p}\mathop{\text{Res}}_{s\to p} \\
&= \mathop{\text{Res}}_{s\to p}\mathop{\text{Res}}_{r\to q} + \mathop{\text{Res}}_{s\to \tilde{q}^j}\mathop{\text{Res}}_{r\to p^i} + \sum_{j\neq 0}\mathop{\text{Res}}_{s\to \tilde{q}^j}\mathop{\text{Res}}_{\tilde{r}^j\to \tilde{q}^j} + \sum_{j\neq 0}\mathop{\text{Res}}_{s\to \tilde{q}^j}\mathop{\text{Res}}_{\tilde{r}^j\to s} \\
&\quad + \sum_{i\neq 0}\mathop{\text{Res}}_{s\to p}\mathop{\text{Res}}_{r^i\to p} + \sum_{i\neq 0}\mathop{\text{Res}}_{s\to p}\mathop{\text{Res}}_{r^i\to s} \\
&= \mathop{\text{Res}}_{s\to p,\tilde{q}^j}\mathop{\text{Res}}_{r\to q,p^i} + \sum_{j\neq 0}\mathop{\text{Res}}_{s\to \tilde{q}^j}\mathop{\text{Res}}_{\tilde{r}^j\to s} + \sum_{i\neq 0}\mathop{\text{Res}}_{s\to p}\mathop{\text{Res}}_{r^i\to s}.
\end{aligned} \tag{3 - 45}$$

The last term does not contribute because the integrant is regular when $r^i \to s$, thus

$$\begin{aligned}
&\frac{\widehat{H}^{(g)}_{k,l}(p,q;\mathbf{p_K};\mathbf{q_L})}{\mathcal{E}(x(p),y(q))} \\
&= \frac{\check{H}^{(g)}_{k,l}(p,q;\mathbf{p_K};\mathbf{q_L})}{\mathcal{E}(x(p),y(q))} + \sum_{i=1}^{d_2} \frac{\mathcal{G}^{(g)}_{k,l}(p,p^i;\mathbf{p_K};\mathbf{q_L})}{(y(q)-y(p))(y(q)-y(p^i))H^{(0)}_{0,0}(p,p^i)dx(p)^2} \\
&+\sum_{j\neq 0}\ \text{Res}_{s\to\tilde{q}^j}\ \text{Res}_{\tilde{r}^j\to s}\ \frac{\mathcal{G}^{(g)}_{k,l}(s,r;\mathbf{p_K};\mathbf{q_L})}{(y(q)-y(r))(x(s)-x(p))(y(r)-y(s))(x(p)-x(q))H^{(0)}_{0,0}(s,r)} \\
&= \frac{\check{H}^{(g)}(p,y(q))}{\mathcal{E}(x(p),y(q))} + \sum_{i=1}^{d_2} \frac{\mathcal{G}^{(g)}_{k,l}(p,p^i;\mathbf{p_K};\mathbf{q_L})}{(y(q)-y(p))(y(q)-y(p^i))H^{(0)}_{0,0}(p,p^i)dx(p)^2} \\
&+\sum_{j=1}^{d_1} \frac{\mathcal{G}^{(g)}_{k,l}(\tilde{q}^j,q;\mathbf{p_K};\mathbf{q_L})}{(x(p)-x(q))(x(\tilde{q}^j)-x(p))H^{(0)}_{0,0}(\tilde{q}^j,q)dy(q)^2} \\
&= \frac{\check{H}^{(g)}(p,y(q))}{\mathcal{E}(x(p),y(q))} \\
&+\sum_{i=1}^{d_2} \frac{(y(p)-y(p^i))}{(y(q)-y(p))(y(q)-y(p^i))}\frac{\tilde{f}^{(g)}_{k,l}(p^i;\mathbf{p_K}|\mathbf{q_L})-\check{W}^{(g)}_{k,l+1}(\mathbf{p_K}|\mathbf{q_L},p^i)}{dx(p)} \\
&+\sum_{j=1}^{d_1} \frac{(x(q)-x(\tilde{q}^j))}{(x(p)-x(q))(x(p)-x(\tilde{q}^j))}\frac{f^{(g)}_{k,l}(\tilde{q}^j|\mathbf{p_K};\mathbf{q_L})-\hat{W}^{(g)}_{k+1,l}(\tilde{q}^j,\mathbf{p_K}|\mathbf{q_L})}{dy(q)}
\end{aligned} \tag{3-46}$$

Notice from Eq. (3-42), that

$$g^{(g)}_{k,l}(s;\mathbf{p_K}|\mathbf{q_L}) \quad := \quad \tilde{f}^{(g)}_{k,l}(s;\mathbf{p_K}|\mathbf{q_L}) - \check{W}^{(g)}_{k,l+1}(\mathbf{p_K}|\mathbf{q_L},s)$$

$$= \qquad -f^{(g)}_{k,l}(s;\mathbf{p_K}|\mathbf{q_L}) + \hat{W}^{(g)}_{k+1,l}(s,\mathbf{p_K}|\mathbf{q_L}) \qquad (3-47)$$

is a holomorphic 1-form in s, i.e. it has no poles. We have:

$$\begin{aligned}
& \frac{\widehat{H}^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L})}{\mathcal{E}(x(p),y(q))} - \frac{\check{H}^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L})}{\mathcal{E}(x(p),y(q))} \\
= & \sum_{i=1}^{d_2} \frac{(y(p)-y(p^i))}{(y(q)-y(p))(y(q)-y(p^i))}\,\frac{g^{(g)}_{k,l}(p^i;\mathbf{p_K}|\mathbf{q_L})}{dx(p)} \\
& -\sum_{j=1}^{d_1} \frac{(x(q)-x(\tilde{q}^j))}{(x(p)-x(q))(x(p)-x(\tilde{q}^j))}\,\frac{g^{(g)}_{k,l}(\tilde{q}^j;\mathbf{p_K}|\mathbf{q_L})}{dy(q)} \\
= & \sum_{i=1}^{d_2} \operatorname*{Res}_{s\to p^i} \frac{(y(p)-y(s))}{(y(q)-y(p))(y(q)-y(s))}\,\frac{g^{(g)}_{k,l}(s;\mathbf{p_K}|\mathbf{q_L})}{(x(s)-x(p))} \\
& -\sum_{j=1}^{d_1} \operatorname*{Res}_{s\to \tilde{q}^j} \frac{(x(q)-x(s))}{(x(p)-x(q))(x(p)-x(s))}\,\frac{g^{(g)}_{k,l}(s;\mathbf{p_K}|\mathbf{q_L})}{(y(s)-y(q))} \\
= & \sum_{i=0}^{d_2} \operatorname*{Res}_{s\to p^i} \Big(\frac{(x(q)-x(s))}{(x(p)-x(q))} + \frac{(y(p)-y(s))}{(y(q)-y(p))}\Big)\,\frac{g^{(g)}_{k,l}(s;\mathbf{p_K}|\mathbf{q_L})}{(x(s)-x(p))(y(q)-y(s))} \\
= & \sum_{i=0}^{d_2} \operatorname*{Res}_{s\to p^i} \Big(\frac{1}{(x(p)-x(q))(y(q)-y(s))} - \frac{1}{(x(p)-x(s))(y(q)-y(s))} \\
& + \frac{1}{(x(s)-x(p))(y(q)-y(p))} - \frac{1}{(x(s)-x(p))(y(q)-y(s))}\Big)\, g^{(g)}_{k,l}(s;\mathbf{p_K}|\mathbf{q_L}) \\
= & \sum_{i=0}^{d_2} \operatorname*{Res}_{s\to p^i} \frac{1}{(x(s)-x(p))(y(q)-y(p))}\, g^{(g)}_{k,l}(s;\mathbf{p_K}|\mathbf{q_L}) \\
= & \ 0
\end{aligned} \qquad (3\text{-}48)$$

Therefore $\widehat{H}^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L}) = \check{H}^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L}) = H^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L})$.

- $\widehat{E}^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L}) = \check{E}^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L})$.

We have from Eq. (3-27)

$$\widehat{E}^{(g)}_{k,l}(p,q,\mathbf{p_K}|\mathbf{q_L}) = (x(p)-x(q))(y(p)-y(q))\widehat{H}^{(g)}_{k,l}(p,q,\mathbf{p_K}|\mathbf{q_L}) - \frac{\mathcal{G}^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L})}{dx(p)dy(q)}, \qquad (3\text{-}49)$$

and from Eq. (3-28):

$$\check{E}^{(g)}_{k,l}(p,q,\mathbf{p_K}|\mathbf{q_L}) = (x(p)-x(q))(y(p)-y(q))\check{H}^{(g)}_{k,l}(p,q,\mathbf{p_K}|\mathbf{q_L}) - \frac{\mathcal{G}^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L})}{dx(p)dy(q)}, \qquad (3\text{-}50)$$

so that $\widehat{E}_{k,l}^{(g)} = \check{E}_{k,l}^{(g)}$.

Moreover, one can see from Eq. (3-27) that $\widehat{E}_{k,l}^{(g)}(p,q;\mathbf{p_K}|\mathbf{q_L})$ is a polynomial of $y(q)$ while $\check{E}_{k,l}^{(g)}(p,q;\mathbf{p_K}|\mathbf{q_L})$ is a polynomial of $x(p)$, therefore

$$E_{k,l}^{(g)}(x(p),y(q);\mathbf{p_K}|\mathbf{q_L}) = \widehat{E}_{k,l}^{(g)}(p,q;\mathbf{p_K}|\mathbf{q_L}) = \check{E}_{k,l}^{(g)}(p,q;\mathbf{p_K}|\mathbf{q_L}) \tag{3-51}$$

is a polynomial in two variables.

•$U_{k,l}^{(g)}$ and $\tilde{U}_{k,l}^{(g)}$ are polynomials.

Eq. (3-49), Eq. (3-50), Eq. (3-37) and Eq. (3-38) imply that

$$\begin{aligned}
& E_{k,l}^{(g)}(x(p),y(q);\mathbf{p_K}|\mathbf{q_L}) \\
= \; & (x(p)-x(q))\tilde{U}_{k,l}^{(g)}(x(p),q;\mathbf{p_K}|\mathbf{q_L}) \\
& + \sum_h \sum_{I,J} \frac{\check{W}_{i,j+1}^{(h)}(\mathbf{p_I};\mathbf{q_J},q)\tilde{U}_{k-i,l-j}^{(g-h)}(x(p),q;\mathbf{p_{K/I}}|\mathbf{q_{L/J}})}{dy(q)} \\
& + \frac{\tilde{U}_{k,l+1}^{(g-1)}(x(p),q;\mathbf{p_K}|\mathbf{q_L},q)}{dy(q)} \\
& - \sum_m d_{q_m} \frac{\tilde{U}_{k,l-1}^{(g)}(x(p),q_m;\mathbf{p_K}|\mathbf{q_{L/\{m\}}})}{y(q)-y(q_m)} \\
& - \sum_m d_{p_m} H_{k-1,l}^{(g)}(p_m,q;\mathbf{p_{K/\{m\}}}|\mathbf{q_L})
\end{aligned} \tag{3-52}$$

and

$$\begin{aligned}
& E_{k,l}^{(g)}(x(p),y(q);\mathbf{p_K}|\mathbf{q_L}) \\
= \; & (y(q)-y(p))U_{k,l}^{(g)}(p,y(q);\mathbf{p_K}|\mathbf{q_L}) \\
& + \sum_h \sum_{I,J} \frac{\hat{W}_{i+1,j}^{(h)}(p,\mathbf{p_I};\mathbf{q_J})U_{k-i,l-j}^{(g-h)}(p,y(q);\mathbf{p_{K/I}}|\mathbf{q_{L/J}})}{dx(p)} \\
& + \frac{U_{k+1,l}^{(g-1)}(p,y(q);p,\mathbf{p_K}|\mathbf{q_L})}{dx(p)} \\
& - \sum_m d_{p_m} \frac{U_{k-1,l}^{(g)}(p_m,y(q);\mathbf{p_{K/\{m\}}}|\mathbf{q_L})}{x(p)-x(p_m)} \\
& - \sum_m d_{q_m} H_{k,l-1}^{(g)}(p,q_m;\mathbf{p_K}|\mathbf{q_{L/\{m\}}})
\end{aligned} \tag{3-53}$$

from which (together with the recursion hypothesis), we deduce that $U_{k,l}^{(g)}$ and $\tilde{U}_{k,l}^{(g)}$ are polynomials.

This proves the theorem 3.2. $\square$

Theorem 3.3 *Symmetry of the* $W_{k,l}^{(g)}$*.*

For any k, l, g *we have:*

$$\hat{W}_{k+1,l+1}^{(g-1)}(p, \mathbf{p_K}|\mathbf{q_L}, q) = \check{W}_{k+1,l+1}^{(g-1)}(p, \mathbf{p_K}|\mathbf{q_L}, q) \tag{3-54}$$

proof:

Let us prove it by recursion on $2g + k + l$. Assume we have already proved it for any g', k', l' such that $2g' + k' + l' < 2g + k + l$.

Insert Eq. (3-30) into Eq. (3-53) in order to eliminate the U's, and then insert the result into Eq. (3-49). Most of the terms cancel (in fact the definitions of $J_{k,l}^{(g)}$, $\mathcal{J}_{k,l}^{(g)}$, $G_{k,l}^{(g)}$ were designed for that purpose), and using the recursion hypothesis, the only term left is:

$$\check{W}_{k+1,l+1}^{(g-1)}(p, \mathbf{p_K}|\mathbf{q_L}, q) = \frac{1}{2}\left(\check{W}_{k+1,l+1}^{(g-1)}(p, \mathbf{p_K}|\mathbf{q_L}, q) + \hat{W}_{k+1,l+1}^{(g-1)}(p, \mathbf{p_K}|\mathbf{q_L}, q)\right) \tag{3-55}$$

which proves the theorem. $\square$

Corollary 3.1 $\hat{W}_{k,l}^{(g)}(\mathbf{p_K}|\mathbf{q_L}) = \check{W}_{k,l}^{(g)}(\mathbf{p_K}|\mathbf{q_L})$ *is a symmetric function of its variables* $p_1, \dots, p_k$*, and a symmetric function of its variables* $q_1, \dots, q_l$*.*

proof:

It is clear from the definitions that $\check{W}_{k,l}^{(g)}(\mathbf{p_K}|\mathbf{q_L})$ is a symmetric function of its variables $p_1, \dots, p_k$, and that $\hat{W}_{k,l}^{(g)}(\mathbf{p_K}|\mathbf{q_L})$ is a symmetric function of its variables $q_1, \dots, q_l$. $\square$

Now, we prove the following theorem:

Theorem 3.4

$$\hat{W}_{k,0}^{(g)}(\mathbf{p_K}|) = \hat{W}_{k}^{(g)}(\mathbf{p_K}) \tag{3-56}$$

and

$$\check{W}_{0,l}^{(g)}(|\mathbf{q_L}) = \check{W}_{l}^{(g)}(\mathbf{q_L}). \tag{3-57}$$

proof:

Write Eq. (3-53) for l=0:

$$\begin{aligned}
& E_{k,0}^{(g)}(x(p), y(q); \mathbf{p_K}) \\
= \quad & (y(q) - y(p))U_{k,0}^{(g)}(p, y(q); \mathbf{p_K}) \\
& + \sum_h \sum_I \frac{\hat{W}_{i+1,0}^{(h)}(p, \mathbf{p_I})U_{k-i,0}^{(g-h)}(p, y(q); \mathbf{p_{K/I}})}{dx(p)} \\
& + \frac{U_{k+1,0}^{(g-1)}(p, y(q); p, \mathbf{p_K})}{dx(p)} - \sum_m d_{p_m} \frac{U_{k-1,0}^{(g)}(p_m, y(q); \mathbf{p_{K/\{m\}}})}{x(p) - x(p_m)}.
\end{aligned}$$

(3 − 58)

Using Lemma B.2, we obtain:

$$\hat{W}^{(g)}_{k,0}(\mathbf{p_K}|) = \hat{W}^{(g)}_{k}(\mathbf{p_K}) \tag{3-59}$$

The other equality is obtained by writing Eq. (3-52) for $k = 0$ and exchanging the roles of x and y in the Lemma B.2.

□

Theorem 3.5

$$\hat{W}^{(g)}_{k+1,l}(p, \mathbf{p_K}|\mathbf{q_L}) + \check{W}^{(g)}_{k,l+1}(\mathbf{p_K}|\mathbf{q_L}, p) = d_p \frac{A^{(g)}_{k,l}(p; \mathbf{p_K}|\mathbf{q_L})}{dx(p)dy(p)} \tag{3-60}$$

where $A^{(g)}_{k,l}(p; \mathbf{p_K}|\mathbf{q_L})$ *has at most simple poles when* $p \to \alpha$.

proof:

From Eq. (3-42), it is easy to see that all contour integrals of $\hat{W}^{(g)}_{k+1,l}(p, \mathbf{p_K}|\mathbf{q_L}) + \check{W}^{(g)}_{k,l+1}(\mathbf{p_K}|\mathbf{q_L}, p)$ are vanishing, and thus it is the differential of some function.

The fact that $A^{(g)}_{k,l}(p; \mathbf{p_K}|\mathbf{q_L})$ has at most simple poles when $p \to \alpha$, follows from lemma 3.1.

□

Theorem 3.6

$$\mathop{\mathrm{Res}}_{p\to\alpha} x(p)y(p)\hat{W}^{(g)}_{k+1,l}(p, \mathbf{p_K}|\mathbf{q_L}) = 0, \tag{3-61}$$

$$\mathop{\mathrm{Res}}_{p\to\alpha} x(p)y(p)\hat{W}^{(g)}_{k,l+1}(\mathbf{p_K}|\mathbf{q_L}, p) = 0. \tag{3-62}$$

proof:

By definition:

$$\hat{W}^{(g)}_{k+1,l}(p, \mathbf{p_K}|\mathbf{q_L}) = \mathop{\mathrm{Res}}_{s\to\mathbf{a},\mathbf{q_L}} dS_{s,o}(p)\, f^{(g)}_{k,l}(s; \mathbf{p_K}|\mathbf{q_L}) \tag{3-63}$$

and we have:

$$\begin{aligned}
& \mathop{\mathrm{Res}}_{p\to\alpha} x(p)y(p)\hat{W}^{(g)}_{k+1,l}(p, \mathbf{p_K}|\mathbf{q_L}) \\
= \quad & \mathop{\mathrm{Res}}_{p\to\alpha} \mathop{\mathrm{Res}}_{s\to\mathbf{a},\mathbf{q_L}} x(p)y(p)dS_{s,o}(p)\, f^{(g)}_{k,l}(s; \mathbf{p_K}|\mathbf{q_L}) \\
= \quad & \mathop{\mathrm{Res}}_{s\to\mathbf{a},\mathbf{q_L}} \mathop{\mathrm{Res}}_{p\to\alpha} x(p)y(p)dS_{s,o}(p)\, f^{(g)}_{k,l}(s; \mathbf{p_K}|\mathbf{q_L}) \\
= \quad & -\mathop{\mathrm{Res}}_{s\to\mathbf{a},\mathbf{q_L}} (x(s)y(s) - x(o)y(o))\, f^{(g)}_{k,l}(s; \mathbf{p_K}|\mathbf{q_L})
\end{aligned}$$

(3 − 64)

since $f^{(g)}_{k,l}$ vanishes near the poles of ydx to order at least $\deg ydx - 1$, the expression above has no other poles than $\mathbf{a}, \mathbf{q_L}$, and thus the total residue is zero.

□

Theorem 3.7 *For any k,l,g such that $k+l+g \leq 1$, one has*

$$\begin{aligned} \mathrm{Res}_{p\to \mathbf{a},\mathbf{q_L}} \Phi(p)\hat{W}^{(g)}_{k+1,l}(p,\mathbf{p_K}|\mathbf{q_L}) &= \mathrm{Res}_{q\to \mathbf{b},\mathbf{p_K}} \Psi(q)\check{W}^{(g)}_{k,l+1}(\mathbf{p_K}|\mathbf{q_L},q) \\ &= (2-2g-k-l)\hat{W}^{(g)}_{k,l}(\mathbf{p_K}|\mathbf{q_L}). \end{aligned} \quad (3\text{-}65)$$

proof:

We have:

$$\begin{aligned} & \mathop{\mathrm{Res}}_{p\to \mathbf{a},\mathbf{q_L}} \Phi(p)\hat{W}^{(g)}_{k+1,l}(p,\mathbf{p_K}|\mathbf{q_L}) - \mathop{\mathrm{Res}}_{p\to \mathbf{b},\mathbf{p_K}} \Psi(p)\check{W}^{(g)}_{k,l+1}(\mathbf{p_K}|\mathbf{q_L},p) \\ = \; & \mathop{\mathrm{Res}}_{p\to \mathbf{a},\mathbf{q_L}} x(p)y(p)\hat{W}^{(g)}_{k+1,l}(p,\mathbf{p_K}|\mathbf{q_L}) - \mathop{\mathrm{Res}}_{p\to \mathbf{a},\mathbf{q_L}} \Psi(p)\hat{W}^{(g)}_{k+1,l}(p,\mathbf{p_K}|\mathbf{q_L}) \\ & \quad - \mathop{\mathrm{Res}}_{p\to \mathbf{b},\mathbf{p_K}} \Psi(p)\check{W}^{(g)}_{k,l+1}(\mathbf{p_K}|\mathbf{q_L},p) \\ = \; & - \mathop{\mathrm{Res}}_{p\to \mathbf{a},\mathbf{b},\mathbf{p_K},\mathbf{q_L}} \Psi(p)(\hat{W}^{(g)}_{k+1,l}(p,\mathbf{p_K}|\mathbf{q_L}) + \check{W}^{(g)}_{k,l+1}(\mathbf{p_K}|\mathbf{q_L},p)) \\ = \; & \mathop{\mathrm{Res}}_{p\to \mathbf{a},\mathbf{b},\mathbf{p_K},\mathbf{q_L}} x(p)dy(p)\frac{A^{(g)}_{k,l}(p;\mathbf{p_K}|\mathbf{q_L})}{dx(p)dy(p)} \\ = \; & -\mathop{\mathrm{Res}}_{p\to \alpha} x(p)dy(p)\frac{A^{(g)}_{k,l}(p;\mathbf{p_K}|\mathbf{q_L})}{dx(p)dy(p)} \\ = \; & 0. \end{aligned} \quad (3\text{-}66)$$

The fact that $\mathrm{Res}_{p\to \mathbf{a},\mathbf{q_L}} \Phi(p)\hat{W}^{(g)}_{k+1,l}(p,\mathbf{p_K}|\mathbf{q_L}) = (2-2g-k-l)\hat{W}^{(g)}_{k,l}(\mathbf{p_K}|\mathbf{q_L})$, can be proved by recursion on $2g+k+l$ and using corolary 3.1.

□

This allows to prove our main theorem:

Theorem 3.8 *The $F^{(g)}$'s are symmetric under the exchange $x \leftrightarrow y$:*

$$\boxed{\hat{F}^{(g)} = \check{F}^{(g)}} \quad (3\text{-}67)$$

proof:

Indeed, we have:

$$(2-2g)\hat{F}^{(g)} = \mathop{\mathrm{Res}}_{\mathbf{a}} \Phi(p)\hat{W}^{(g)}_{1,0}(p) \qquad , \qquad (2-2g)\check{F}^{(g)} = \mathop{\mathrm{Res}}_{\mathbf{b}} \Psi(p)\check{W}^{(g)}_{0,1}(p). \quad (3\text{-}68)$$

□

3.4 Additional properties

The following theorem relates H and W:

Theorem 3.9 *We have:*

$$\hat{W}^{(g)}_{k+1,l}(p,\mathbf{p_K}|\mathbf{q_L}) = \mathop{\mathrm{Res}}_{q\to\alpha} \frac{H^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L})}{H^{(0)}_{0,0}(p,q)}dy(q) \quad (3\text{-}69)$$

$$\check{W}^{(g)}_{k,l+1}(\mathbf{p_K}|\mathbf{q_L},q) = \operatorname*{Res}_{p\to\alpha} \frac{H^{(g)}_{k,l}(p,q;\mathbf{p_K}|\mathbf{q_L})}{H^{(0)}_{0,0}(p,q)} dx(p). \tag{3-70}$$

proof:

Multiply equation 3-30 by $dx(p)dy(q)/(y(q)-y(p))H^{(0)}_{0,0}(p,q)$ and take the residues at $q \to \alpha$.

□

Remark 3.3 This theorem was expected from the matrix model property that

$$\operatorname{tr} \frac{1}{x-M_1}\frac{1}{y-M_2} \to \frac{1}{x} \operatorname{tr} \frac{1}{y-M_2} \tag{3-71}$$

when $x \to \infty$.

4 Conclusion

In this article, we have proved the $x \leftrightarrow y$ symmetry which was announced in [21]. This symmetry has many applications, for instance in [21] it was used to recover the $(p,q) \leftrightarrow (q,p)$ duality of minimal models [30], or to give a very short proof that Kontsevitch integral indeed depends only on odd times and satisfies KdV hierarchy [26].

In addition we have shown how to compute some family of mixed correlation functions of the 2-matrix model.

This could open the route to some matrix model approach to the understanding of boundary conformal field theory in higher genus. In a forthcoming article, we shall introduce a similar algebraic geometry method to compute all possible mixed correlation functions [23].

This work also raises many questions, and calls the following prospects:

• It would be interesting to see what the $H_{k,l}$ and $W_{k,l}$ correspond to for other matrix models (e.g. Kontsevitch's integral, chain of matrices), although we may guess that they also correspond to mixed traces expectation values in those cases.

• More interesting would be to understand what the $H^{(g)}_{k,l}$ and $W^{(g)}_{k,l}$ compute in algebraic geometry. Those should correspond to "volume" or "intersection numbers of some moduli spaces" ?

Acknowledgements

We would like to thank Michel Bergère and Aleix Prats Ferrer for fruitful discussions on this subject. This work is partly supported by the Enigma European network MRT-CT-2004-5652, by the ANR project Géométrie et intégrabilité en physique mathématique

ANR-05-BLAN-0029-01, by the Enrage European network MRTN-CT-2004-005616, by the European Science foundation through the Misgam program, by the French and Japaneese governments through PAI Sakura, by the Quebec government with the FQRNT.

Appendix A Spectral curve

We recall that the curve $\mathcal{E}(x,y)$, called the classical spectral curve, is given by a polynomial of the form:

$$\mathcal{E}(x,y) = \sum_{j=0}^{d_2+1} \mathcal{E}_j(x) y^j \tag{1-1}$$

We define the "quantum spectral curve" as the formal power series:

$$\mathcal{E}_N(x,y) = \sum_g N^{-2g}\, \mathcal{E}^{(g)}(x,y) \tag{1-2}$$

where

$$\mathcal{E}^{(g)}(x,y) = \mathcal{E}_{d_2+1}(x) \sum_{r=1}^{d_2} \sum_{J_1 \cup \ldots \cup J_r = K} \sum_{g_1,\ldots,g_r} \delta_{\sum_l (g_l + |J_l| - 1), g} \prod_{l=1}^{r} \tilde{W}^{(g_l)}_{|J_l|}(p^{J_l}) \tag{1-3}$$

with

$$K = \{1, \ldots, d_2\} \tag{1-4}$$

and

$$\tilde{W}^{(g)}_k(p_K) := W^{(g)}_k(p_K) + \delta_{k,1}\delta_{g,0}(y - Y(p_1)) \tag{1-5}$$

where $W^{(g)}_k(p_K)$ is the meromorphic form defined in [21] for the curve $\mathcal{E}(x,y)$.

Lemma A.1 *For any g, $\mathcal{E}^{(g)}(x,y)$ is a polynomial in x and y, whose degrees are at most those of $\mathcal{E}$.*

proof:

It is clear that $\mathcal{E}_N(x,y)$ is a polynomial in y, and a rational function of x. Let us prove that $\mathcal{E}^{(g)}(x,y)$ is indeed a polynomial in x for $g \geq 1$. The coefficient of y^k in $\mathcal{E}^{(g)}(x,y)$ is:

$$\begin{aligned} &\frac{\mathcal{E}^{(g)}_k(x)}{\mathcal{E}_{d_2+1}(x)} \\ &= \sum_{J_0 \subset K, |J_0| = k} \prod_{j \in J_0} y(p^j) \sum_{r=1}^{d_2-k} \sum_{J_1 \cup \ldots \cup J_r = K/J_0} \sum_{g_1,\ldots,g_r} \delta_{\sum_l (g_l + |J_l| - 1), g} \prod_{l=1}^{r} W_{|J_l|}{}^{(g_l)}(p^{J_l}) \end{aligned} \tag{1-6}$$

First, notice that the product of W's can have poles only at branch-points, and the product of y's can have poles only at poles of y. The poles of y which are not poles of x, are killed by the prefactor $\mathcal{E}_{d_2+1}(x)$, as they are in the classical curve $\mathcal{E}(x,y)$. Let us consider the poles at a branch-point a. The only terms which might diverge at $p \to a$ are of either of the following forms

- $(W^{(h)}_{1+|J|}(p,p^J)+W^{(h)}_{1+|J|}(\bar{p},p^J))\times(\text{reg})$ where reg means a term with no poles at $p\to a$. This term is regular because of theorem 4.4 in [21].

- or $(W^{(g_1)}_{1+|J_1|}(p,p^{J_1})W^{(h-g_1)}_{1+|J|-|J_1|}(\bar{p},p^{J/J_1})+W^{(h-1)}_{2+|J|}(p,\bar{p},p^J))\times(\text{reg})$ again, this expression is regular when $p\to a$, because of theorems 4.4 and 4.5 in [21].

Thus, we have proved that $\mathcal{E}^{(g)}_k(x)$ is a rational function of x whose only poles are the poles of x, i.e. it is a polynomial in x.

Consider a pole ∞_x of x, the behavior of $\mathcal{E}^{(g)}(x(p),y(p))$ when $p\to\infty_x$ is at most that of $\sum_{J_0\subset K}\prod_{j\in J_0}y(p^j)$. Notice that J_0 cannot be equal to K itself, because the product of the corresponding W's vanishes (it contains no term), and $|J_0|$ cannot be equal to $|K|-1$, because the prefactor vanishes due to theorem 4.4 in [21]. Thus, $|J_0|\leq|K|-2$, which implies that $\mathcal{E}^{(g)}(x(p),y(p))dx(p)$ has a pole of degree at most that of $\mathcal{E}_y(x(p),y(p))$, i.e. $\mathcal{E}^{(g)}(x(p),y(p))$ is contained in the Newton's polytope of $\mathcal{E}(x,y)$. This means that

$$\frac{\mathcal{E}^{(g)}(x(p),y(p))}{\mathcal{E}_y(x(p),y(p))}dx(p) \tag{1-7}$$

is a holomorphic differential.

$\square$

Appendix B Lemma: unicity of the solution of loop equations

Lemma B.2 *The system of equations:*

$$\begin{aligned}
& E^{(g)}_k(x(p),y(q);\mathbf{p_K}) \\
= \quad & (y(q)-y(p))U^{(g)}_k(p,y(q);\mathbf{p_K}) \\
& +\sum_h\sum_I \frac{W^{(h)}_{i+1}(p,\mathbf{p_I})U^{(g-h)}_{k-i}(p,y(q);\mathbf{p_{K/I}})}{dx(p)} \\
& +\frac{U^{(g-1)}_{k+1}(p,y(q);p,\mathbf{p_K})}{dx(p)}-\sum_m d_{p_m}\frac{U^{(g)}_{k-1}(p_m,y(q);\mathbf{p_{K/\{m\}}})}{x(p)-x(p_m)}
\end{aligned} \tag{2-1}$$

where:

- *if $2g+k>2$, $W^{(g)}_{k+1}(p,\mathbf{p_K})$ has poles only at branchpoints in any of its variables, and vanishing $\mathcal{A}$-cycle integrals,*

- *$E^{(g)}_k(x(p),y(q);\mathbf{p_K})$ is a polynomial in $x(p)$ of degree at most d_1-1, and a polynomials in $y(q)$ of degree at most d_2-1,*

- $U_k^{(g)}(p,y(q);\mathbf{p_K})$ *is a polynomials in* $y(q)$ *of degree at most* d_2-1,

has a unique solution.

This solution is such that

$$W_k^{(g)}(\mathbf{p_K}) = \hat{W}_k^{(g)}(\mathbf{p_K}). \tag{2-2}$$

Proof of the Lemma:

Unicity:

We prove it by recursion on $2g+k$. Assume it is already proved for any g',k' such that $2g'+k<2g+k$.

At $p=q$, Eq. (2-1) gives:

$$\begin{aligned} W_{k+1}^{(g)}(p,\mathbf{p_K}) \quad = \quad & \frac{E_k^{(g)}(x(p),y(p);\mathbf{p_K})dx(p)}{U_0^{(0)}(p,y(p))} \\ & -\sum_h\sum_I \frac{W_{i+1}^{(h)}(p,\mathbf{p_I})U_{k-i}^{(g-h)}(p,y(p);\mathbf{p_{K/I}})}{U_0^{(0)}(p,y(p))} \\ & -\frac{U_{k+1}^{(g-1)}(p,y(p);p,\mathbf{p_K})}{U_0^{(0)}(p,y(p))} + \sum_m d_{p_m}\frac{U_{k-1}^{(g)}(p_m,y(p);\mathbf{p_{K/\{m\}}})dx(p)}{(x(p)-x(p_m))U_0^{(0)}(p,y(p))}. \end{aligned} \tag{2-3}$$

Then write Cauchy residue formula:

$$W_{k+1}^{(g)}(p,\mathbf{p_K}) = -\mathop{\mathrm{Res}}_{r\to p} dS_{r,o}(p)\; W_{k+1}^{(g)}(r,\mathbf{p_K}). \tag{2-4}$$

Since we know the poles of $W_{k+1}^{(g)}(p,\mathbf{p_K})$ and its $\mathcal{A}$-cycle integrals, we may move the integration contour using Riemann's bilinear identity and get:

$$W_{k+1}^{(g)}(p,\mathbf{p_K}) = \mathop{\mathrm{Res}}_{r\to \mathbf{a}} dS_{r,o}(p)\; W_{k+1}^{(g)}(r,\mathbf{p_K}). \tag{2-5}$$

Now, we replace $W_{k+1}^{(g)}(r,\mathbf{p_K})$ by its value in Eq. (2-3). We see that the term $\frac{E_k^{(g)}(x(r),y(r);\mathbf{p_K})dx(r)}{U_0^{(0)}(r,y(r))}$ has no pole at the branchpoints and does not contribute to the residue, and similarly the las term of Eq. (2-3) does not contribute to the residue. We get:

$$\begin{aligned} W_{k+1}^{(g)}(p,\mathbf{p_K}) \quad = \quad & -\mathop{\mathrm{Res}}_{r\to \mathbf{a}} \frac{dS_{r,o}(p)}{U_0^{(0)}(p,y(p))}\Big(U_{k+1}^{(g-1)}(r,y(r);p,\mathbf{p_K}) \\ & +\sum_h\sum_I W_{i+1}^{(h)}(r,\mathbf{p_I})U_{k-i}^{(g-h)}(r,y(r);\mathbf{p_{K/I}})\Big). \end{aligned} \tag{2-6}$$

Since all the terms in the RHS are already known from the recursion hypothesis, this determines $W_{k+1}^{(g)}(p,\mathbf{p_K})$ uniquely. Then, we write Eq. (2-1) for $p=\tilde{q}^j$ with $j=1,\dots,d_1$:

$$E_k^{(g)}(x(\tilde{q}^j),y(q);\mathbf{p_K})$$

$$
\begin{aligned}
= \quad & \sum_h \sum_I \frac{W^{(h)}_{i+1}(\tilde{q}^j, \mathbf{p_I}) U^{(g-h)}_{k-i}(\tilde{q}^j, y(q); \mathbf{p_{K/I}})}{dx(\tilde{q}^j)} \\
& + \frac{U^{(g-1)}_{k+1}(\tilde{q}^j, y(q); \tilde{q}^j, \mathbf{p_K})}{dx(\tilde{q}^j)} - \sum_m d_{p_m} \frac{U^{(g)}_{k-1}(p_m, y(q); \mathbf{p_{K/\{m\}}})}{x(\tilde{q}^j) - x(p_m)}
\end{aligned}
\tag{2-7}
$$

since all terms in the RHS are uniquely determined, so is the LHS. And since $E^{(g)}_k(x(p), y(q); \mathbf{p_K})$ is a polynomial in $x(p)$ of degree $d_1 - 1$ and we know its value in d_1 points, then $E^{(g)}_k(x(p), y(q); \mathbf{p_K})$ is uniquely determined.

Then, using Eq. (2-1) once again, we uniquely determine $U^{(g)}_k(p, y(q); \mathbf{p_K})$.

This proves the unicity for g and k.

Existence:

Start from the meromorphic form $W^{(g)}_k(p_K)$ defined in [21] for the curve $\mathcal{E}(x, y)$, and define:

$$\tilde{W}^{(g)}_k(p_K) := W^{(g)}_k(p_K)/dx(p_K) + \delta_{k,1}\delta_{g,0}(y - y(p_1)) \tag{2-8}$$

Then, let $K_0 = \{0, 1, \ldots, d_2\} \cup K$ and $K_1 = \{1, \ldots, d_2\} \cup K$, and define:

$$\mathcal{E}^{(g)}_k(x(p^0), y; p_K) = \mathcal{E}_{d_2+1}(x) \sum_{r=1}^{d_2+1+k} \sum_{J_1 \cup \ldots \cup J_r = K_0} \sum_{g_1, \ldots, g_r} \delta_{\sum_l (g_l + |J_l| - 1), g} \prod_{l=1}^{r} \tilde{W}^{(g_l)}_{|J_l|}(p^{J_l}) \tag{2-9}$$

and:

$$U^{(g)}_k(p^0, y; p_K) = \mathcal{E}_{d_2+1}(x) \sum_{r=1}^{d_2+k} \sum_{J_1 \cup \ldots \cup J_r = K_1} \sum_{g_1, \ldots, g_r} \delta_{\sum_l (g_l + |J_l| - 1), g} \prod_{l=1}^{r} \tilde{W}^{(g_l)}_{|J_l|}(p^{J_l}). \tag{2-10}$$

It is clear that both $\mathcal{E}^{(g)}_k(x, y; p_K)$ and $U^{(g)}_k(p, y; p_K)$ are polynomials in y of degree at most $d_2 - 1$. Following the same line as in lemma A.1, it is easy to get that $\mathcal{E}^{(g)}_k(x, y; p_K)$ is also a polynomial in x of degree at most $d_1 - 1$.

Therefore, the functions $\mathcal{E}^{(g)}_k(x, y; p_K)$, $U^{(g)}_k(p, y; p_K)$ and $W^{(g)}_k(p_K)$ obey the requirements of lemma B.2, and eq.2-1 is clearly satisfied from the definitions of $\mathcal{E}^{(g)}_k(x, y; p_K)$ and $U^{(g)}_k(p, y; p_K)$. Thus, we have found an explicit solution of the system of lemma B.2, which proves the existence.

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